Homework 20 - Convergence of Series with Positive Terms

1) a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\quad f(x) = \frac{1}{x^2+1}$ is continuous ✓

positive ✓

decreasing ✓

on $[1, \infty)$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^2+1}\,dx$$

$$= \lim_{R\to\infty} \tan^{-1}x \Big|_1^R$$

$$= \lim_{R\to\infty} \left[\tan^{-1}R - \tan^{-1}1\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

$\int_1^{\infty} \frac{1}{x^2+1}\,dx$ is convergent, so $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent by Integral Test

b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ $\quad f(x) = \frac{1}{x(\ln x)^2}$ is continuous ✓

positive ✓

decreasing ✓

on $[2, \infty)$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^2}\,dx$$

$u = \ln x \quad du = \frac{1}{x}$

$$= \lim_{R\to\infty} \int_{\ln 2}^{\ln R} u^{-2}\,du$$

$$= \lim_{R\to\infty} -\frac{1}{u} \Big|_{\ln 2}^{\ln R}$$

$$= \lim_{R\to\infty} -\frac{1}{\ln R} + \frac{1}{\ln 2}$$

$$= \frac{1}{\ln 2}$$

$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ is convergent so $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent by Integral Test

2 a) $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

$0 \leq \frac{1}{n2^n} \leq \frac{1}{2^n}$ for all n

$\sum \frac{1}{2^n}$ is a convergent geometric series

$|r| = \frac{1}{2} < 1$

$\therefore$ $\sum \frac{1}{n2^n}$ is convergent by CT

b) $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}+2^n}$

$0 \leq \frac{1}{n^{1/2}+2^n} \leq \frac{1}{2^n}$ for all n

$\sum \frac{1}{2^n}$ is a convergent geometric series

$|r| = \frac{1}{2} < 1$

$\therefore$ $\sum \frac{1}{n^{1/2}+2^n}$ is convergent by CT

c) $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

$0 \leq \frac{\sin^2 k}{k^2} \leq \frac{1}{k^2}$

$\sum \frac{1}{k^2}$ is a convergent p-series

$p = 2 > 1$

$\therefore$ $\sum \frac{\sin^2 k}{k^2}$ is convergent by CT

d) $\sum_{k=2}^{\infty} \frac{k^{1/3}}{k^{5/4}-k}$

$\frac{k^{1/3}}{k^{5/4}-k} \geq \frac{k^{1/3}}{k^{5/4}} = \frac{1}{k^{11/12}} \geq 0$

$\sum \frac{1}{k^{11/12}}$ is a divergent p-series

$p = \frac{11}{12} \not> 1$

$\therefore$ $\sum_{k=2}^{\infty} \frac{k^{1/3}}{k^{5/4}-k}$ is divergent by CT

3) a) $\sum_{n=2}^{\infty} \frac{n^2}{n^4-1}$ $a_n = \frac{n^2}{n^4-1}$ $b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$ positive terms

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^4-1} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^4}{n^4-1}$$

$$= 1 \neq 0 \text{ , finite}$$

$\sum b_n$ is a convergent p-series, $p=2>1$

$\therefore \sum a_n$ is convergent by LCT

b) $\sum_{n=3}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$ $a_n = \frac{3n+5}{n(n-1)(n-2)}$ $b_n = \frac{3n}{n^3} = \frac{3}{n^2}$

positive terms

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n+5}{n(n-1)(n-2)} \cdot \frac{n^2}{3} = 1 \neq 0, \text{ finite}$$

$\sum \frac{3}{n^2}$ is a convergent p-series, $p=2>1$

$\therefore \sum a_n$ is convergent by LCT