Homework 25 - Power Series

1 a) $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

geometric $|r| = \frac{|x|}{3}$

convergent for $\frac{|x|}{3} < 1$

$|x| < 3$

$R = 3$, does not converge at endpoints

b) $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ $a_n = \frac{x^n}{n3^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n\,3^n}{|x|^n}$$

$$= \frac{|x|^{n+1}}{|x|^n} \cdot \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}}$$

$$= |x| \left(\frac{n}{n+1}\right) \cdot \frac{1}{3}$$

$$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}|x|$$

convergent for $\frac{|x|}{3} < 1$

$|x| < 3$ $R = 3$

Check $x = -3$

$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n}\right)$

convergent

Check $x = 3$

$\sum_{n=1}^{\infty} \frac{1}{n}$

divergent

$[-3, 3)$

converges at one endpoint but not the other

c) $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$ $\quad a_n = \frac{|x|^n}{n^2 3^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{|x|^n}$$

$$= \left|\frac{x^{n+1}}{x^n}\right| \cdot \frac{n^2}{(n+1)^2} \cdot \frac{3^n}{3^{n+1}}$$

$$= |x| \cdot \left(\frac{n}{n+1}\right)^2 \left(\frac{1}{3}\right)$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{3}$$

convergent for $\frac{|x|}{3} < 1$

$|x| < 3 \quad R = 3$

check $x = -3$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

convergent

check $x = 3$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$

convergent

$[-3, 3]$

converges at both endpoints

2 a) $\sum_{n=0}^{\infty} n x^n$ $\quad a_n = n x^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1\,|x|^{n+1}}{n\,|x|^n}$$

$$= \left(\frac{n+1}{n}\right)|x|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$$

convergent for $|x| < 1$ $\quad R = 1$

$(-1, 1)$

Check $x = -1$

$\sum_{n=0}^{\infty} n(-1)^n$

Diverges

Check $x = 1$

$\sum_{n=0}^{\infty} n$

Diverges

b) $\sum_{n=0}^{\infty} n(x-3)^n$ $\quad a_n = n(x-3)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)|x-3|^{n+1}}{n\,|x-3|^n}$$

$$= \left(\frac{n+1}{n}\right)|x-3|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-3|$$

convergent for $|x-3| < 1$ $\quad R = 1$

Check $x-3 = -1$
$x = 2$

$\sum_{n=0}^{\infty} n(-1)^n$

Diverges

Check $x-3 = 1$
$x = 4$

$\sum_{n=0}^{\infty} n$

Diverges

$(2, 4)$