

Series Practice Solutions

$$\sum \left(1 + \frac{1}{n}\right) 2^{-n} = \left(\sum 2^{-n} + \sum \frac{1}{n} 2^{-n}\right), \text{ if both series are convergent}$$

$$\sum 2^{-n} \text{ is a convergent geometric series, } |r| = \frac{1}{2} < 1$$

Show that $\sum \frac{1}{n} 2^{-n}$ is convergent.

1) $\frac{1}{n} 2^{-n} < 2^{-n}$ for $n \geq 1$ (both positive for all n)

$$\sum \frac{1}{n} 2^{-n} \text{ is convergent by the comparison test.}$$

So $\sum \left(1 + \frac{1}{n}\right) 2^{-n}$ is convergent.

It is absolutely convergent because the terms are all positive.

$$\sum \sin\left(\left(\frac{2n+1}{2}\right)\pi\right) n^{-\frac{1}{2}} = \sum (-1)^n n^{-\frac{1}{2}}$$

$$\left|(-1)^n n^{-\frac{1}{2}}\right| = n^{-\frac{1}{2}}$$

$$(n+1)^{-\frac{1}{2}} < n^{-\frac{1}{2}} \text{ for all } n$$

2) $\lim_{n \rightarrow \infty} n^{-\frac{1}{2}} = 0$

$\sum (-1)^n n^{-\frac{1}{2}}$ is convergent by the Alternating Series Test

$$\sum n^{-\frac{1}{2}} \text{ is a divergent } p\text{-series, } p = \frac{1}{2} < 1$$

So $\sum (-1)^n n^{-\frac{1}{2}}$ is conditionally convergent.

Use LCT to compare $\sum \frac{\ln n}{n+2}$ to $\sum \frac{1}{n}$ (both series have positive terms)

$$\lim_{n \rightarrow \infty} \left[\frac{\frac{\ln n}{n+2}}{\frac{1}{n}} \right] = \lim_{n \rightarrow \infty} \left[\ln n \left(\frac{n}{n+2} \right) \right] = \infty$$

3) So $\frac{\ln n}{n+2}$ is of higher order than $\frac{1}{n}$

$\sum \frac{1}{n}$ is a divergent p-series, $p = 1$ (not greater than 1)

So $\sum \frac{\ln n}{n+2}$ is divergent by the LCT.

$$\lim_{n \rightarrow \infty} \ln \left(\frac{n}{3n+1} \right) = \ln \left(\frac{1}{3} \right)$$

4) $\lim_{n \rightarrow \infty} (-1)^n \ln \left(\frac{n}{3n+1} \right)$ does not exist (and is not equal to 0)

So $\sum (-1)^n \ln \left(\frac{n}{3n+1} \right)$ is divergent by the Test for Divergence

$$\sum \frac{(-2)^n}{n^2}$$

Use Ratio Test.

$$a_n = \frac{(-2)^n}{n^2}$$

5) $|a_n| = \frac{2^n}{n^2}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}}{(n+1)^2} \frac{n^2}{2^n} = \frac{2^{n+1}}{2^n} \frac{n^2}{(n+1)^2} = 2 \left(\frac{n}{n+1} \right)^2$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 2 > 1$$

So the series is divergent by the Ratio Test.

$$\sum \frac{1}{9^n} (2n)!$$

Use Ratio Test.

$$a_n = \frac{(2n)!}{9^n}$$

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{(2(n+1))!}{9^{n+1}} \cdot \frac{9^n}{(2n)!} \\ &= \frac{(2n+2)!}{(2n)!} \cdot \frac{9^n}{9^{n+1}} \\ &= \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \cdot \frac{1}{9} \\ &= \frac{(2n+2)(2n+1)}{9} \end{aligned}$$

$$6) \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty > 1$$

So the series is divergent by the Ratio Test.

$$\sum \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{5^n n!}$$

Use Ratio Test.

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{5^n n!}$$

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{1 \cdot 3 \cdot 5 \cdots (2(n+1)-1)}{5^{n+1} (n+1)!} \cdot \frac{5^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{n!}{(n+1)!} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{n!}{(n+1)!} \\ &= \frac{2n+1}{1} \cdot \frac{1}{5} \cdot \frac{1}{n+1} \\ &= \frac{2n+1}{5(n+1)} \end{aligned}$$

$$7) \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{5} < 1$$

So the series is absolutely convergent by the Ratio Test.