Homework 1
Solutions

Problem 1

A captain boasts that his ship guns have a maximum range of 64 km. (He had a few drinks.) Each shell leaves the gun at horizontal velocity $u\cos\alpha$ and vertical velocity $u\sin\alpha$. $u > 0$ is the velocity; the angle α is between $0°$ and $90°$. For simplicity we do not[1] consider any physical or geometrical effects other than gravitation, with downward gravitational acceleration $g = 10\ m/s^2$.
(a) Find formulas for the vertical velocity $w(t)$ and height $y(t)$ as functions of time t after firing.
(b) How many seconds until the shell comes back to ground? (Find a formula.)
(c) Find a formula for the horizontal distance $x(t)$.
(d) How far does the shell fly? (Find a formula.)
(e) Show why the angle $\alpha = 45°$ yields the biggest distance (regardless of $u > 0$).
(f) At that angle, what u is needed to reach the claimed maximum range?

Solution (a) See section 1.2. Derivative of velocity is acceleration: $\dot{w}(t) = -g$. Hence $w(t) = w(0) - gt$. $w(0)$ is the initial vertical velocity, so $w(t) = u\sin\alpha - gt$. Derivative of height is vertical velocity: $\dot{y}(t) = w(t) = u\sin\alpha - gt$. So $y(t) = u\sin\alpha t - \frac{1}{2}gt^2$.

(b) At ground means $0 = y(t) = t(u\sin\alpha - \frac{1}{2}gt)$. Either $t = 0$ (initial time) or

$$t = 2u\sin\alpha/g =: T$$

(final time).

(c) $x(t) = u\cos\alpha t$.

(d) $x(T) = 2u^2\sin\alpha\cos\alpha/g$.

(e) Maximize $x(T)$ over $\alpha \in [0, \pi/2]$. For $\alpha = 0$ and $\alpha = \pi/2$ we get $x(T) = 0$. Interior: derivative wrt α is

$$2u^2(\cos^2\alpha - \sin^2\alpha)/g$$

In a maximum it must be zero. That means $|\cos\alpha| = |\sin\alpha|$, hence $\alpha = \pi/4$ (other solutions are not in the range 0 to $\pi/2$). With this α the distance is

$$u^2/g$$

(f)

$$64000\ m = u^2/(10\ m/s^2)$$

means

$$u = \sqrt{640000}\ m/s = 800\ m/s$$

(This is over twice the speed of sound which is 340 m/s at standard conditions, so having neglected the air resistance we cannot expect the result to be accurate.)

Problem 2

(a) Draw a slope field for the differential equation

$$\frac{dy}{dx} = xy$$

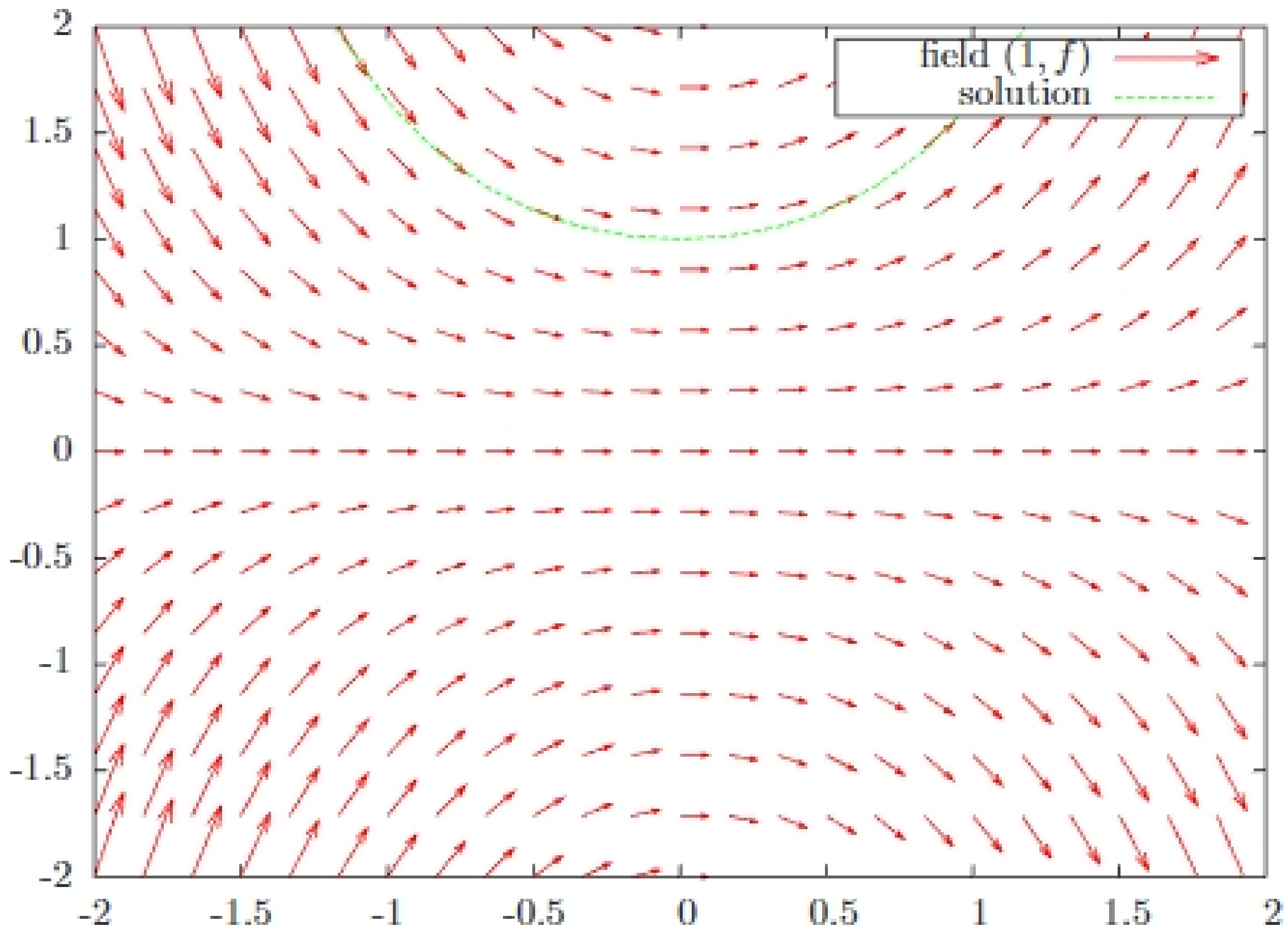
[1] In particular, ignore air resistance, earth curvature, earth rotation, ship motion, ...

over the range $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$. Use it to sketch the solution for $y(0) = 1$.

(b) Check that $y(x) = -e^{x^2/2}$ is a solution for initial condition $y(0) = -1$.

(c) Is that the only solution for that initial condition? Or could there be another one? Consider Theorem 1 in section 1.3 and check its assumptions carefully!

Solution (a) Diagram:



(b) Take the derivative:

$$\frac{dy}{dx} = \frac{d}{dx}(-e^{x^2/2}) = -xe^{x^2/2}$$

Compare:

$$xy = x \cdot (-e^{x^2/2})$$

Indeed they are the same.

(c) $f(x, y) = xy$ and $\partial f/\partial y = x$ are continuous on any rectangle R, including those containing $(a, b) = (0, -1)$. Say $R = [-1, 1] \times [-1, 1]$. Hence by section 1.3 Theorem 1, there *exists* a *unique* solution $y(x)$ defined on $(-\delta, \delta)$ for some $\delta > 0$. There cannot be more than one solution.