

Homework 2

Solutions

Problem 1

Which of the two equations

$$\frac{dy}{dx} + y^2 \sin x = 0$$

$$(1+x)\frac{dy}{dx} + y = \cos x$$

is linear? Solve that one (ignore the other one), with initial condition $y(0) = 1$.

Solution The second one is linear. Integrating factor:

$$G = \exp \int P(x)dx = \exp \int \frac{1}{1+x} dx = \exp(C + \log|1+x|) = e^C |1+x|$$

Since the initial condition is at $x = 0$, where $1+x > 0$, we may take $|1+x| = 1+x$. Moreover, we only need that $dG/dx = PG$, which is true for any choice of e^C ; it is convenient to take $e^C = 1$. So:

$$G = 1+x$$

Then

$$\frac{d}{dx}(Gy) = GQ$$

ie

$$\frac{d}{dx}((1+x)y) = (1+x)\frac{1}{1+x} \cos x = \cos x$$

Integrate:

$$(1+x)y(x) - (1+0)\underbrace{y(0)}_{=1} = \int_0^x \cos x dx = \sin x - \sin 0 = \sin x$$

so the solution is

$$y(x) = \frac{1}{1+x}(1 + \sin x)$$

Problem 2

Find the solution of

$$\frac{dy}{dx} = y^2(1-2x) \quad , \quad y(0) = -\frac{1}{2}$$

Find the largest x interval containing 0 on which y is well-defined. Sketch the graph of y over that interval.

Solution Separation:

$$-y^{-2}dy = (2x - 1)dx$$

Integrate:

$$y^{-1} = x^2 - x + C$$

Initial condition:

$$-2 = y^{-1}(0) = 0^2 - 0 + C = C$$

So the solution is

$$y(x) = \frac{1}{x^2 - x - 2}$$

Problem 3

Find the general solution of the differential equation

$$x^3 \frac{dy}{dx} + 4x^2 y = 1 + x^2$$

and then find the particular solution satisfying $y(1) = 1$.

Solution

$$\frac{dy}{dx} + \underbrace{\frac{4}{x}}_{=P} y = \underbrace{x^{-3} + x^{-1}}_{=Q}$$

This is a linear equation. Integrating factor:

$$G = \exp \int P(x)dx = \exp \int \frac{4}{x} dx = \exp(c + 4 \log |x|) = e^c |x|^4$$

Again, the initial condition is at $x = 1$, and $|x| = x$ for $x > 0$; moreover any nonzero constant ensures $dG/dx = PG$, so we may take

$$G = x^4$$

(other choices work as well, just more writing). Now

$$\frac{d}{dx}(Gy) = GQ$$

ie

$$\frac{d}{dx}(x^4 y) = x^4(x^{-3} + x^{-1}) = x + x^3$$

Integrate:

$$x^4 y(x) - \underbrace{1^4 y(1)}_{=1} = \int_1^x x + x^3 dx = \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{2} - \frac{1}{4}$$

so the solution is

$$y(x) = x^{-4}\left(1 + \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{3}{4}\right) = \frac{1}{4}x^{-4} + \frac{1}{2}x^{-2} + \frac{1}{4} [= \frac{1}{4}(x^{-2} + 1)^2]$$

Problem 4

Consider

$$\frac{dy}{dt} = \underbrace{(1 + \cos^2(t^{100}y^{543} + \exp t))}_{=f(t,y)} \operatorname{sign} y, \quad \operatorname{sign} y = \begin{cases} 1, & y > 0 \\ 0, & y = 0 \\ -1, & y < 0 \end{cases}$$

[Do not lose time trying to find a solution formula; the equation is too complicated.]

- (a) f is continuous in which (t, y) ?
- (b) What, if anything, does the Theorem in section 1.3 tell you about existence of solutions for initial condition $y(0) = 1$?
- (c) What, if anything, does the Theorem say about existence if the initial condition is $y(0) = 0$?

Solution (a) $\operatorname{sign} y$ is continuous at any $y \neq 0$, but not at $y = 0$. The other factor in f is continuous at all (t, y) , but also positive. Hence f is continuous at $y \neq 0$, but not at $y = 0$.

(b) f is continuous near the initial point $(t, y) = (0, 1)$. So is $\partial f / \partial y$ (since $\operatorname{sign} y$ is constant on each side of 0). [Although the book version of the Theorem requires checking $\partial f / \partial y$ as well, that is necessary only for *uniqueness*, not for *existence*.] So the Theorem guarantees existence of a solution $y : (-\delta, \delta) \rightarrow \mathbb{R}$ with $y(0) = 1$, where $\delta > 0$. (The Theorem does not yield a formula for y or δ .)

(c) f is not continuous in the initial point $(t, y) = (0, 0)$, so the Theorem does not say anything. [You might have spotted, however, that $f(t, 0) = 0$ for all t , so $y(t) = 0$ for all t defines a solution. You were not required to see this.]