Homework 5
Solutions

Problem 1

In a mass-spring system with friction and external forcing, the position x of the mass is governed by the differential equation

$$x'' + 2x' + 10x = 2\cos(2t) + 4\sin(2t).$$

Determine the steady periodic solution of this equation, and express it in the form

$$x(t) = C\cos(\omega t - \alpha)$$

where C and ω are positive and α is in the range $[0, 2\pi)$.

Solution The homogeneous equation is

$$x'' + 2x' + 10x = 0$$

with characteristic equation

$$0 = r^2 + 2r + 10 = (r+1)^2 - 1^2 + 10 = (r+1)^2 + 9 \quad \Leftrightarrow \quad r = -1 \pm 3i$$

The corresponding particular solutions are

$$y_1(t) = e^{-t}\cos(3t) \quad , \quad y_2(t) = e^{-t}\sin(3t)$$

with general solution

$$y_H(t) = e^{-t}(c_1\cos(3t) + c_2\sin(3t))$$

To find a particular solution of the *in*homogeneous problem we use undetermined coefficients. Try $x(t) = \cos(2t)$:

$$\begin{aligned} x' &= -2\sin(2t) \\ x'' &= -4\cos(2t) \\ x'' + 2x' + 10x &= -4\cos(2t) - 4\sin(2t) + 10\cos(2t) \end{aligned}$$

Try $x(t) = \sin(2t)$:

$$\begin{aligned} x' &= 2\cos(2t) \\ x'' &= -4\sin(2t) \\ x'' + 2x' + 10x &= -4\sin(2t) + 4\cos(2t) + 10\sin(2t) \end{aligned}$$

So try a linear combination $x(t) = C_1\cos(2t) + C_2\sin(2t)$: by superposition that is

$$\begin{aligned} x'' + 2x' + 10x &= C_1\big(-4\cos(2t) - 4\sin(2t) + 10\cos(2t)\big) + C_2\big(-4\sin(2t) + 4\cos(2t) + 10\sin(2t)\big) \\ &= (6C_2 - 4C_1)\sin(2t) + (4C_2 + 6C_1)\cos(2t) \end{aligned}$$

Looking at the desired right-hand side, we seek to solve

$$\begin{aligned} 6C_2 - 4C_1 &= 4 \\ 4C_2 + 6C_1 &= 2 \end{aligned}$$

which is a linear system that is solved in the usual way:

$$C_2 = -\frac{1}{13} \quad , \quad C_1 = \frac{8}{13}$$

The resulting particular solution is

$$y_I(t) = \frac{1}{13}\big(8\sin(2t) - \cos(2t)\big)$$

Then the general solution of the inhomogeneous problem is

$$y(t) = y_H(t) + y_I(t) = \frac{1}{13}\big(8\sin(2t) - \cos(2t)\big) + e^{-t}(c_1\cos(3t) + c_2\sin(3t))$$

The initial data determines c_1, c_2. However, regardless what their values are, the e^{-t} factor causes that term to decay exponentially as $t \to \infty$. Hence the limiting steady periodic solution is exactly y_I.

Now we put it into a more convenient form:

$$\begin{aligned} y_I(t) &= -\frac{1}{13}\cos(2t) + \frac{8}{13}\sin(2t) \\ &\stackrel{!}{=} C\cos(\omega t - \alpha) = C\cos\alpha\cos(\omega t) + C\sin\alpha\sin(\omega t) \end{aligned}$$

Matching these two we see that $\omega = 2$ and

$$\frac{8/13}{-1/13} = \frac{C\sin\alpha}{C\cos\alpha} = \tan\alpha$$

Since $(-1/13, 8/13)$ is in the *left* halfplane, while arctan yields only angles in the right halfplane, we consider the *antipodal* point $(1/13, -8/13)$ instead. Since it is antipodal, we get to its angle by subtracting π: $\alpha - \pi$.

$$-8 = \tan(\alpha - \pi)$$

$$\alpha = \pi + \arctan(-8) = \pi - \arctan 8 = 1.695...$$

As discussed in class,

$$C = \sqrt{(\frac{1}{13})^2 + (\frac{8}{13})^2} = \sqrt{\frac{5}{13}}$$

Hence

$$y_I(t) = \sqrt{\frac{5}{13}}\cos(2t - \pi + \arctan 8)$$

is the desired solution.

Problem 2

Find the general solution of the differential equation

$$4y'' + 4y' + 5y = 0$$

Solution Characteristic equation:

$$0 = 4r^2 + 4r + 5 = (2r+1)^2 - 1^2 + 5 = (2r+1)^2 + 4 \quad \Leftrightarrow \quad 2r + 1 = \pm 2i \quad \Leftrightarrow \quad r = -\frac{1}{2} \pm i$$

This is a conjugate pair of roots, each is a simple root. So the particular solutions are

$$y_1(t) = e^{-\frac{1}{2}t}\cos t \quad , \quad y_2(t) = e^{-\frac{1}{2}t}\sin t$$

and the general solution is

$$y(t) = e^{-\frac{1}{2}t}(c_1 \cos t + c_2 \sin t)$$

Problem 3

Find all (complex) roots of the polynomials $z^3 - 8i$ and $z^4 + 16$. [Hint: Use polar coordinates.] For full credit use $\sin, \cos$ and simplify them as far as possible.

Solution One method is to spot some special roots, such as $z = -2i$ for $z^3 = 8i$, and to perform polynomial division. But the general method is as follows:

$$z^4 = -16 = 16\exp(i\pi)$$

$$z_k = \sqrt[4]{16}\exp(i\frac{\pi + 2\pi k}{4})$$

$$z_0 = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = 2(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \sqrt{2}(1+i)$$

$$z_1 = 2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) = 2(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \sqrt{2}(-1+i)$$

We could continue, but since the equation $z^4 + 16 = 0$ has real coefficients we immediately see that the conjugates $\sqrt{2}(1-i)$ and $\sqrt{2}(-1-i)$ are also solutions.

$$z^3 = 8i = 8\exp(i\frac{\pi}{2})$$

Solutions:

$$z_k = \sqrt[3]{8}\exp(i\frac{\pi/2 + 2\pi k}{3}) \quad (k = 0, 1, 2)$$

$$z_0 = 2\exp\frac{i\pi}{6} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = \sqrt{3} + i$$

$$z_1 = 2\exp\frac{5i\pi}{6} = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = 2(-\frac{\sqrt{3}}{2} + i\frac{1}{2}) = -\sqrt{3} + i$$

$$z_2 = 2\exp\frac{3i\pi}{2} = -2i$$

Here the conjugates are not roots; the equation $z^3 - 8i = 0$ has a non-real coefficient $-8i$.

Problem 4

Find the general solution of the differential equation

$$y'''' - 2y''' + 3y'' - 2y' + y = 0.$$

[Hint: Expand $(r^2 - r + 1)^2$ first.]