Homework 6
Solutions

Problem 1

The homogeneous differential equation

$$t^2y'' + ty' - y = 0$$

has general solution

$$y(t) = At + B/t$$

where A and B are arbitrary constants. Find a particular solution of the non-homogeneous equation

$$t^2y'' + ty' - y = t$$

[Hint: use variation of parameters, and be sure to account for the fact that the coefficient of y'' is not 1.]

Solution Divide by t^2:

$$y'' + t^{-1}y' - t^{-2}y = t^{-1} = f$$

Wronskian: with $y_1 = t$ and $y_2 = t^{-1}$,

$$W = y_1y_2' - y_1'y_2 = t(-t^{-2}) - 1t^{-1} = -2t^{-1}$$

$$u_1 = \int \frac{-y_2}{W} f\,dt = \int \frac{-t^{-1}}{-2t^{-1}} t^{-1}dt = \frac{1}{2}\log|t| + C_1$$

$$u_2 = \int \frac{y_1}{W} f\,dt = \int \frac{t}{-2t^{-1}} t^{-1}dt = -\frac{1}{4}t^2 + C_2$$

$$y = u_1y_1 + u_2y_2 = \tilde{C}_1t + \tilde{C}_2t^{-1} + \frac{1}{2}t\log|t|$$

where $\tilde{C}_1, \tilde{C}_2$ are arbitrary constants. For example

$$y = -\frac{1}{4}t + \frac{1}{2}t\log|t|$$

Problem 2

(a) Sketch the direction field for the autonomous system

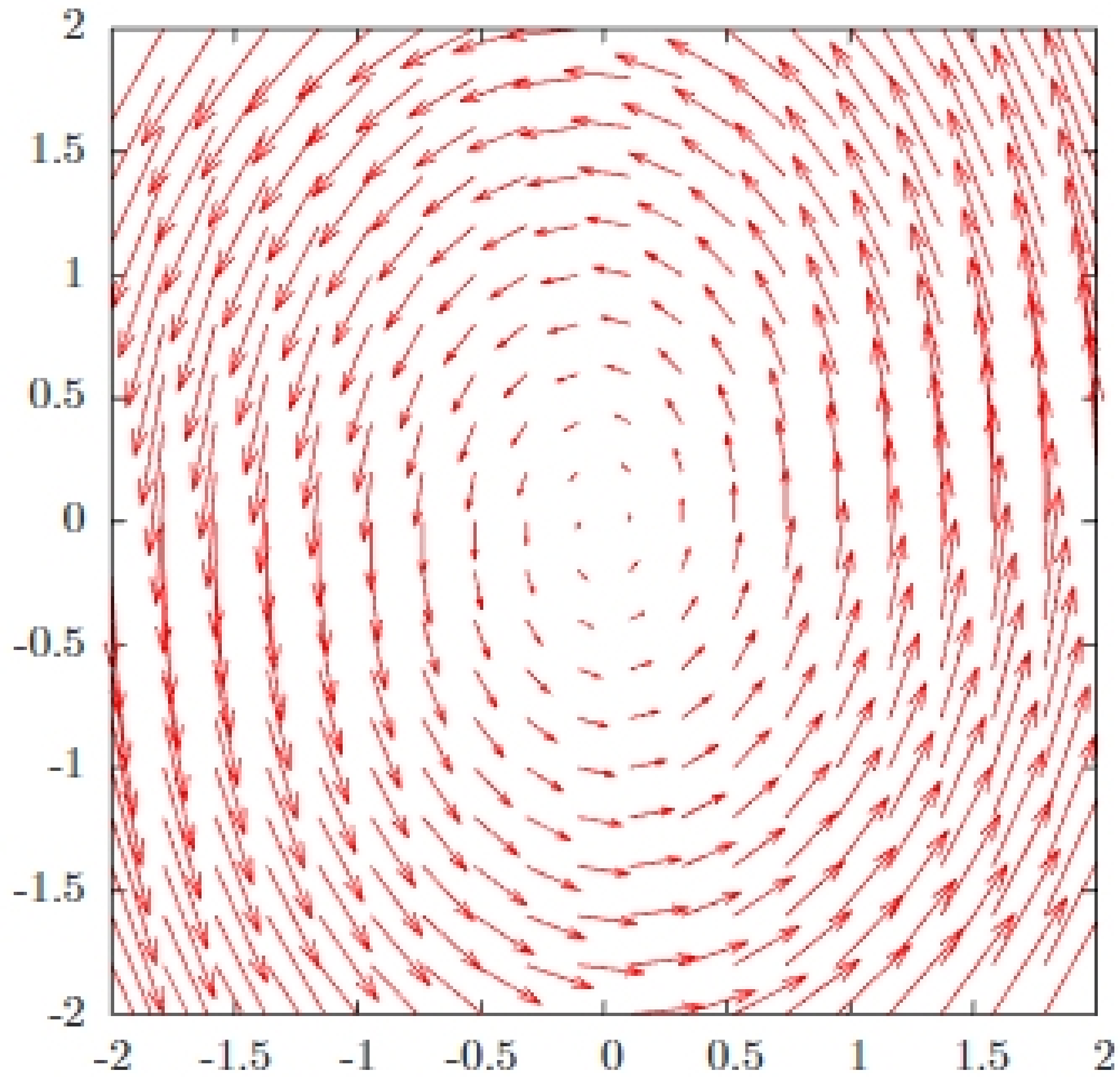
$$x' = -y \quad , \quad y' = 2x$$

Then sketch the trajectory for initial value $(x, y) = (1, 0)$.

(b) Does the trajectory return to the initial value? To decide, calculate the time derivative g' of

$$g = 2x^2 + y^2$$

Then sketch the isolines of g and try to answer the question.



Solution (a)

(b) The computer-generated direction field above suggests the trajectories spiral away from the origin. But diagrams can be misleading.

$$g' = (2x^2 + y^2)' = 4xx' + 2yy' = -4xy + 4yx = 0$$

That means g is constant along each trajectory. The curves of constant g are ellipses

$$y = \pm\sqrt{C - 2x^2}$$

(for constants $C = 2x^2_{\max} > 0$) which are closed curves. So the trajectories return to the initial value.

Problem 3

Given the higher-order equation

$$x''' + 3x'' + x = 0$$

Since there are up to three derivatives, we introduce the new variables

$$y_0 = x \quad , \quad y_1 = x' \quad , \quad y_2 = x'' \quad , \quad y_3 = x'''$$

This is unnecessary (the recommended way is to use y_0, y_1, y_2, but not y_3), but can be carried out. Find the correct right-hand side for

$$\begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} \quad ... \quad \\ ... \\ ... \\ ... \end{bmatrix}$$

using only $y_0, ..., y_3$, without derivatives.

Solution Differentiate the equation once to obtain

$$x'''' + 3x''' + x' = 0$$

Then

$$\begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} x' \\ x'' \\ x''' \\ x'''' \end{bmatrix} = \begin{bmatrix} x' \\ x'' \\ x''' \\ -3x''' - x' \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ -3y_3 - y_1 \end{bmatrix}$$