

## Homework 7 Solutions

### Problem 1

Solve

$$x'(t) = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} x(t)$$

using the eigenvalue method, for initial condition

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For full credit, simplify all solutions to be purely real.

**Solution** (For the sake of illustration, this solution is a bit more elaborate than necessary.)

Characteristic polynomial:

$$0 = \det \begin{pmatrix} 3-z & -2 \\ 4 & -1-z \end{pmatrix} = (3-z)(-1-z) - (-2) \cdot 4 = z^2 - 2z + 5 = (z-1)^2 + 4 \Leftrightarrow z = 1 \pm 2i$$

So the eigenvalues are

$$\lambda_1 = 1 + 2i \quad , \quad \lambda_2 = 1 - 2i$$

The second is the conjugate of the first, which is expected since the matrix is real. Equation for an eigenvector  $v_1$  for  $\lambda_1 = 1 + 2i$ :

$$\begin{bmatrix} 3 - (1 + 2i) & -2 \\ 4 & -1 - (1 + 2i) \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First row:

$$(2 - 2i)v_{11} - 2v_{21} = 0$$

We pick  $v_{11} = 1$ , then  $v_{21} = 1 - i$ . The eigenvector is

$$v_1 = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$$

An eigenvector  $v_2$  for  $\lambda_2$  can be obtained analogously, but it is faster to take the conjugate of  $v_1$ :

$$v_2 = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}$$

We obtain particular solutions

$$\begin{aligned} x_1(t) &= e^{\lambda_1 t} v_1 = e^{(1+2i)t} \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = e^t e^{2it} \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} \\ &= e^t (\cos(2t) + i \sin(2t)) \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = e^t \begin{bmatrix} \cos(2t) + i \sin(2t) \\ \cos(2t) + \sin(2t) + i(\sin(2t) - \cos(2t)) \end{bmatrix} \end{aligned}$$

and

$$x_2(t) = e^{\lambda_2 t} v_2 = \dots = e^t \begin{bmatrix} \cos(2t) - i \sin(2t) \\ \cos(2t) + \sin(2t) + i(-\sin(2t) + \cos(2t)) \end{bmatrix}$$

$x_2$  is the conjugate of  $x_1$  which is not accidental of course. Now the real part of  $x_1$  is the linear combination  $\frac{1}{2}(x_1 + x_2)$  which is

$$\hat{x}_1 = e^t \begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix}$$

and the imaginary part is  $(x_1 - x_2)/2i$  which is

$$\hat{x}_2 e^t \begin{bmatrix} -\sin(2t) \\ -\sin(2t) + \cos(2t) \end{bmatrix}$$

The Wronskian of these two is

$$\begin{aligned} W(t) &= \det \begin{bmatrix} e^t \cos(2t) & -e^t \sin(2t) \\ e^t(\cos(2t) + \sin(2t)) & e^t(-\sin(2t) + \cos(2t)) \end{bmatrix} \\ &= e^t \cos(2t) \cdot e^t(-\sin(2t) + \cos(2t)) - (-e^t \sin(2t)) \cdot e^t(\cos(2t) + \sin(2t)) \\ &= e^{2t}(\cos(2t)^2 + \sin(2t)^2) = e^{2t} \end{aligned}$$

which is nonzero for all  $t$ . So  $\hat{x}_1, \hat{x}_2$  are linearly independent.

General solution:

$$x(t) = c_1 \hat{x}_1(t) + c_2 \hat{x}_2(t) = \begin{bmatrix} e^t(c_1 \cos(2t) - c_2 \sin(2t)) \\ e^t(c_1(\cos(2t) + \sin(2t)) + c_2(\cos(2t) - \sin(2t))) \end{bmatrix}$$

To satisfy the initial condition, solve

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x(0) = \begin{bmatrix} c_1 \\ c_1 + c_2 \end{bmatrix}$$

with solution  $c_1 = 1$  and  $c_2 = -1$ . Then

$$x(t) = \begin{bmatrix} e^t(\cos(2t) + \sin(2t)) \\ e^t 2 \sin(2t) \end{bmatrix}$$

## Problem 2

Find the general solution of

$$x'(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} x(t)$$

using the eigenvalue method.

**Solution** We calculate the characteristic polynomial by expansion across column 1:

$$0 = \det \begin{bmatrix} -z & 0 & 1 \\ 0 & 2-z & 0 \\ 1 & 0 & -z \end{bmatrix} = (-z)^2(2-z) - 1 \cdot (2-z) \cdot 1 = (2-z)(z^2 - 1) = (2-z)(z-1)(z+1)$$

The eigenvalues are

$$\lambda_1 = 1 \quad , \quad \lambda_2 = -1 \quad , \quad \lambda_3 = 2$$

Eigenvector  $v_1$  for  $\lambda_1 = 1$ :

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} = \begin{bmatrix} v_{31} - v_{11} \\ v_{21} \\ v_{11} - v_{31} \end{bmatrix}$$

We find  $v_{21} = 0$  and  $v_{11} = v_{31}$ ; one nonzero solution is

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

In analogous fashion we find eigenvectors

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

General solution:

$$x(t) = \begin{bmatrix} c_1 e^t + c_2 e^{-t} \\ c_3 e^{2t} \\ c_1 e^t - c_2 e^{-t} \end{bmatrix}$$

### Problem 3

Calculate the eigenvalues of

$$A = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}$$

for arbitrary  $a$ . Then sketch the direction fields of  $y' = Ay$  for  $a = -1$ ,  $a = 0$  and  $a = 1$ . In each case, for which initial values  $y(0)$  do the solutions  $y(t)$

- (a) converge to  $(0, 0)$ , or
  - (b) converge to infinity (meaning  $|y(t)| \rightarrow \infty$ ), or
  - (c) neither,
- as  $t \rightarrow \infty$ ?

**Solution**

$$0 = \det \begin{bmatrix} a - z & 1 \\ -1 & a - z \end{bmatrix} = (a - z)^2 - 1 \cdot (-1) = (a - z)^2 + 1 \quad \Leftrightarrow \quad z = a \pm i$$

The eigenvalues are  $\lambda_1 = a + i$  and  $\lambda_2 = a - i$ . The general solution is

$$y(t) = e^{at}(c_1 e^{it} v_1 + c_2 e^{-it} v_2)$$