Name ______________________________ **My Class Time** _____ 2250 Midterm 3 [Ver 2, S2010]

Applied Differential Equations 2250

Exam date: Thursday, 22 April, 2010

Instructions: This in-class exam is 50 minutes. Up to 60 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Chapter 5)** Complete all.

(1a) [40%] Write the solution $x(t)$ of

$$x''(t) + 9x(t) = 30\sin(2t), \quad x(0) = x'(0) = 0,$$

as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form**.

Answer:
$x(t) = -4\sin(3t) + 6\sin(2t)$

(1b) [30%] Determine the practical resonance frequency ω for the electric current equation

$$2I'' + 7I' + 100I = 100\omega\cos(\omega t).$$

Answer:
$\omega = 1/\sqrt{LC} = 1/\sqrt{2/100} = \sqrt{50} = 5\sqrt{2}$.

(1c) [30%] Apply the variation of parameters formula (33) in Edwards-Penney,

$$y_p(x) = y_1(x)\left(\int \frac{-y_2(x)f(x)}{W(x)}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(x)}dx\right)$$

to find a particular solution y_p of the equation

$$y''(x) + 2y'(x) + 17y(x) = 4e^{-x}.$$

Expected are all integration details and a shortest expression for $y_p(x)$.

Answer:
$y_p(t) = y_1(x)\int k1(x)dx + y_2(x)\int k2(x)dx$, $f(x) = 4e^{-x}$, $y_1(x) = e^{-x}\cos(4x)$, $y_2(x) = e^{-x}\sin(4x)$, $W(x) = 4e^{-2x}$, $k_1(x) = -y_2(x)f(x)/W(x) = -\sin(4x)$, $k_2(x)f(x) = y_1(x)f(x)/W(x) = \cos(4x)$. Then

$$\begin{aligned} y_p &= y_1(x)\int -\sin(4x)dx + y_2(x)\int\cos(4x)dx \\ &= \frac{1}{4}e^{-x}(\cos^2(4x) + \sin^2(4x)) \\ &= \frac{1}{4}e^{-x}. \end{aligned}$$

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2. **(Chapter 5)** Complete all.

(2a) [60%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 8 with roots $0, 0$, $2, 2$, $2i, -2i$, 7, 7 listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 2xe^x + 3e^{2x} + 4x^2 + 5\cos 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer:

The atoms of $f(x)$ are xe^x, e^{2x}, x^2, $\cos 2x$. Complete the list to 8 atoms, by adding the related atoms, to obtain 5 groups, each group having exactly one base atom: (1) e^x, xe^x, (2) e^{2x}, (3) $1, x, x^2$, (4) $\cos 2x$, (5) $\sin 2x$. The trial solution is a linear combination of 8 atoms, modified by rules to the new list (1) e^x, xe^x,(2) x^2e^{2x}, (3) x^2, x^3, x^4, (4) $x\cos 2x$, (5) $x \sin 2x$. The root 7 of the homogeneous equation is not used in the calculation.

(2b) [40%] Let $f(x) = 3e^x + 4\sin(2x)(\sin(2x) + \cos(2x))$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

Answer:

Because $\sin^2(2x) = (1-\cos 4x)/2$ [identity $\cos 2\theta = 1-2\sin^2\theta$], and $\sin(2x)\cos(2x) = \frac{1}{2}\sin(4x)$, then root $r = 0$ and roots $r = \pm 4i$ are used to produce the second term $4\sin(2x)(\sin(2x) + \cos(2x))$. Total roots: $1, 0, \pm 4i$ with product of the factors $(r-1)(r-0)(r^2+16) = r^4 - r^3 + 16r^2 - 16r$. The DE: $y^{(4)} - y''' + 16y'' - 16y' = 0$.

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3. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(3a) [50%] Display the details of Laplace's method to solve the system for $x(t)$. Don't waste time solving for $y(t)$!

$$x' = 3x + 2y,$$
$$y' = -x,$$
$$x(0) = 4, \quad y(0) = 0.$$

Answer:

The Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ can be written out to find a 2×2 linear system for unknowns $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$. Cramer's rule applies to this system to solve for $\mathcal{L}(x(t)) = \dfrac{4s}{(s-2)(s-1)}$. Then partial fractions and backward table methods determine $x(t) = 8e^{2t} - 4e^{t}$.

(3b) [25%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s + 48}{s^2(s+4)}.$$

Answer:

$$\mathcal{L}(f(t)) = \frac{12}{s^2} + \frac{-1}{s} + \frac{1}{s+4} = \mathcal{L}(-1 + 12t + e^{-4t})$$

(3c) [25%] Solve for $f(t)$, given

$$\frac{d^2}{ds^2}\mathcal{L}(f(t)) = \frac{120}{s^5}.$$

Answer:

Use the s-differentiation theorem and the backward Laplace table to get $(-t)^2 f(t) = 120t^4/4!$ or $f(t) = 5t^2$.

Use this page to start your solution. Attach extra pages as needed, then staple.