Worksheet #7 Solutions

1. $\frac{4}{3^n} = 4 \cdot \frac{1}{3^n} = 4 \cdot \left(\frac{1}{3}\right)^n \leftarrow$ Geometric w/ $r<1$!

So $\boxed{\sum_{n=1}^{\infty} \frac{4}{3^n} \text{ converges}}$. It converges to $\frac{ar^{n_0}}{1-r} = \frac{4\left(\frac{1}{3}\right)^1}{1-\frac{1}{3}}$

$a=4$, $r=\frac{1}{3}$, $n_0=1$

$$\boxed{\sum_{n=1}^{\infty} \frac{4}{3^n} = 2}$$

2. $3^{2-3n} = \frac{3^2}{3^{3n}} = \frac{9}{(3^3)^n} = \frac{9}{(27)^n} = 9 \cdot \left(\frac{1}{27}\right)^n$. $\leftarrow$ Geometric w/ $r<1$!

By the same argument in 1., $\boxed{\sum_{n=3}^{\infty} 3^{2-3n} \text{ converges}}$

It converges to: $\frac{ar^{n_0}}{1-r} = \boxed{\frac{9\left(\frac{1}{27}\right)^3}{1-\frac{1}{27}}}$ $a=9$, $r=\frac{1}{27}$, $n_0=3$.

3. $\frac{2^{3n+1}}{7^{n+100}} = \frac{2^{3n}\,2}{7^n\,7^{100}} = \frac{2}{7^{100}} \cdot \frac{8^n}{7^n} = \frac{2}{7^{100}}\left(\frac{8}{7}\right)^n$

This is geometric w/ $r>1$, so $\boxed{\sum_{n=5}^{\infty} \frac{2^{3n+1}}{7^{n+100}} \text{ diverges}}$

4. $\sum_{n=0}^{\infty} 4(r^2)^n$ converges if $r^2<1 \Rightarrow |r|<1$

In this case, it converges to $\frac{a\tilde{r}^{n_0}}{1-\tilde{r}} = \frac{a(r^2)^0}{1-r^2} = \frac{a}{1-r^2}$

So: $$\boxed{\sum_{n=0}^{\infty} 4r^{2n} \begin{cases} \text{converges to } \frac{a}{1-r^2} \text{ if } |r|<1 \\ \text{diverges, otherwise} \end{cases}}$$

5. $\frac{3^{2n+1}}{10^{n+2}} = \frac{3^{2n}\,3^1}{10^n\,10^2} = \frac{3}{100}\,\frac{9^n}{10^n} = \frac{3}{100}\left(\frac{9}{10}\right)^n$. $\leftarrow$ Geometric w/ $r<1$!

So, $\boxed{\sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^{n+2}} \text{ converges}}$. It converges to: $\frac{ar^{n_0}}{1-r} = \frac{\frac{3}{100}\cancel{\left(\frac{9}{10}\right)^0}}{1-\frac{9}{10}} = \boxed{\frac{3}{10}}$

6. Note: $\frac{(-5)^n}{6^{2n}} = \frac{(-5)^n}{(6^2)^n} = \frac{(-5)^n}{36^n} = \left(-\frac{5}{36}\right)^n \leftarrow$ Geometric w/ $r<1$!

So $\boxed{\sum_{n=0}^{\infty} \frac{(-5)^n}{6^{2n}} \text{ converges}}$ It converges to: $\frac{ar^n}{1-r} = \frac{1\left(-\frac{5}{36}\right)^0}{1-\left(-\frac{5}{36}\right)}$

$= \boxed{\frac{36}{41}}$

7. Note: $4(-1.75)^{n/3} = 4\left[(-1.75)^{1/3}\right]^n \leftarrow$ This is geometric w/ $r=(-1.75)^{1/3}$ so $|r|>1$!

Hence $\boxed{\sum_{n=0}^{\infty} 4(-1.75)^{n/3} \text{ diverges}}$

8. $\sum_{n=1}^{\infty} \frac{2^{3n}}{60000} = \sum_{n=1}^{\infty} \frac{1}{60000} 8^n \leftarrow$ geometric w $r>1$!

$\boxed{\sum_{n=1}^{\infty} \frac{2^{3n}}{60000} \text{ diverges}}$

9. $\sum_{n=3}^{\infty} \frac{2-4n}{6+n}$. Note: $\lim_{n\to\infty} \frac{6-4n}{6+n} = -4 \neq 0$.

$\boxed{\text{The series diverges by the divergence test}}$

10. $\sum_{n=1}^{\infty} \frac{2^n}{n!} \leftarrow \lim_{n\to\infty} \frac{2^n}{n!} = 0$ so div test fails! Use Ratio Test:

Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right|$

$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} \frac{\cancel{2^n}\, 2^1}{\cancel{2^n}} \cdot \frac{\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\ldots}{(n+1)\cancel{(n)}\cancel{(n-1)}\cancel{(n-2)}}$

$= \lim_{n\to\infty} \frac{2}{n+1} = \underline{0}$

$\boxed{\text{The series converges by Ratio Test since } L = 0 < 1.}$

11. $\displaystyle\sum_{n=2}^{\infty} \frac{(n^2+1)\,3^{2n}}{(2n)!}$

Ratio Test:

$$L = \lim_{n\to\infty} \left| \frac{[(n+1)^2+1]\;3^{2(n+1)}}{[2(n+1)]!} \cdot \frac{(2n)!}{(n^2+1)\,3^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2+2n+2}{n^2+1} \cdot \frac{3^{2n+2}}{3^{2n}} \cdot \frac{(2n)!}{(2n+2)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2+2n+2}{n^2+1} \cdot \frac{\cancel{3^{2n}}\,3^2}{\cancel{3^{2n}}} \cdot \frac{\cancel{2n(2n-1)\ldots}}{(2n+2)(2n+1)\cancel{2n(2n-1)\ldots}} \right|$$

$$= \lim_{n\to\infty} \left| \underbrace{\frac{n^2+2n+2}{n^2+1}}_{\downarrow\, 1} \cdot 3^2 \cdot \underbrace{\frac{1}{(2n+2)(2n+1)}}_{\downarrow\, 0} \right|$$

$\underline{L = 0.}$

$\boxed{\displaystyle\sum_{n=2}^{\infty} \frac{(n^2+1)\,3^{2n}}{(2n)!} \text{ converges by the ratio test since } L = 0 < 1}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{2^n}{4^n} + \frac{3^n}{4^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$

Both sums on the RHS are geometric w/ $r < 1$ so the series converges

13. $\displaystyle\lim_{n\to\infty} \frac{6n^2-1}{\sqrt{n^4+7}} = \lim_{n\to\infty} \frac{6n^2-1}{\sqrt{n^4\left(1+\frac{7}{n^4}\right)}} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(6-\frac{1}{n^2}\right)}{\cancel{n^2}\sqrt{1+\frac{7}{n^4}}} = 6.$

Hence, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{6n^2-1}{\sqrt{n^4+7}} \text{ diverges by the divergence test.}}$