Math 199 Mock Exam · 16 Sept, 2014

1. Determine the range of the function $g(x)$, and determine the domain of the function $h(x)$. Write your answers in interval notation.

$$f(x) = \frac{x+2}{x^2-100} \qquad g(x) = \sqrt{x-4} \qquad h = f \circ g$$

The range of $g(x)$ is $[0, \infty)$. Let's consider the domain of h. First, we'll write out h:

$$h(x) = \frac{\sqrt{x-4}+2}{\sqrt{x-4}^2-100}.$$

Here are the things that must be true:

(i) $x - 4 \geq 0$

(ii) $\sqrt{x-4}^2 - 100 \neq 0$

For (i), we see $x \geq 4$. For (ii), we calculate:

$$\begin{aligned} \sqrt{x-4}^2 - 100 &\neq 0 \\ \Leftrightarrow \sqrt{x-4}^2 &\neq 100 \\ \Leftrightarrow \sqrt{x-4} &\neq \pm 10 \\ \Leftrightarrow x - 4 &\neq 100 \\ \Leftrightarrow x &\neq 104 \end{aligned}$$

Combining (i) and (ii), we see our domain is all numbers greater than or equal to 4, and not equal to 104. That is:

$$[4, 104) \cup (104, \infty)$$

2. For each of the sequences below, write out a_{10}, a_{20}, and a_{1000}. Then, choose one of them and prove it is strictly increasing, and choose another and prove it is bounded from above. Give the limit of the remaining sequence.

(a) $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ $a_{10} = 1/\sqrt{10}$, $a_{20} = 1/\sqrt{20}$, $a_{1000} = 1/\sqrt{1000}$

(b) $a_n = 15 - \dfrac{1}{n^2}$ $a_{10} = 15 - 1/100$, $a_{20} = 15 - 1/400$, $a_{1000} = 15 - 1/(1000)^2$

(c) $a_n = \left(\dfrac{7}{10}\right)^n$ $a_{10} = (7/10)^{10}$, $a_{20} = (7/10)^{20}$, $a_{1000} = (7/10)^{1000}$

The only sequence that is strictly increasing is (b), so we prove that. Remember, we have to prove that $a_n < a_{n+1}$. To do this, notice:

$$\begin{aligned}
(n+1) &> n \\
\Rightarrow (n+1)^2 &> n^2 \\
\Rightarrow \frac{1}{(n+1)^2} &< \frac{1}{n^2} \\
\Rightarrow -\frac{1}{(n+1)^2} &> -\frac{1}{n^2} \\
\Rightarrow 15 - \frac{1}{(n+1)^2} &> 15 - \frac{1}{n^2} \\
\Rightarrow a_{n+1} &> a_n.
\end{aligned}$$

Now, we need to show that one of the remaining is bounded from above, and find the limit of the other. You can choose which sequence is which. Both have limit 0. I'll prove that (c) is bounded from above.

Note that (c) is strictly decreasing, so its first term, 7/10, is an upper bound. More formally:

$$(7/10)^n \leq (7/10) \text{ whenever } n \geq 1$$

3. True or False: If the terms of a sequence are alternately positive and negative numbers, the sequence cannot possibly be convergent.

   False: it can still converge (but only to 0). Consider $a_n = \dfrac{(-1)^n}{n}$

4. For each of the items at the left, select the item on the right that fits the best. Some items on the right may be used more than once, or not at all.

   (a) $a_n = (-1/2)^n$ (III)
   (b) $a_n = (-2)^n$ (V)
   (c) $(a_n) = (0.9, 0.99, 0.9, 0.99, 0.9, 0.99, \ldots)$(V)
   (d) $a_n = \dfrac{n+1}{1000}$(I)
   (e) $a_n = \dfrac{n+1}{n^2}$(III)
   (f) $a_n = \dfrac{5n+1}{n}$(VI)

   (I) $\lim_{n\to\infty} a_n = \infty$
   (II) $\lim_{n\to\infty} a_n = -\infty$
   (III) $\lim_{n\to\infty} a_n = 0$
   (IV) $\lim_{n\to\infty} a_n = 1$
   (V) a_n is divergent
   (VI) a_n is convergent

5. Consider the sequence generated by $a_n := \dfrac{2n-2}{n+1}$.

   (a) Use the Limit Laws to determine the limit of the sequence. We can only use the limit laws on convergent sequences so we first need to do some rearranging. We multiply the top and bottom by $\frac{1}{n}$ and so we get

   $$a_n = \frac{\frac{1}{n}(2n-2)}{\frac{1}{n}(n+1)} = \frac{2-\frac{2}{n}}{1+\frac{1}{n}}.$$

   We can now use the limit laws and so

   $$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} 2 - 2\lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n}}.$$

We have a theorem that tells us $\lim_{n\to\infty} \frac{1}{n} = 0$ and so using this above, we get

$$\lim_{n\to\infty} a_n = \frac{2 - 2\cdot 0}{1+0} = 2.$$

(b) Determine the value of n for which all subsequent terms of the sequence will be within $\frac{1}{1000}$ of the limit. We want the value of n such that $|a_n - L| < \frac{1}{1000}$. Substituting the formula for a_n and the value of the limit found above, we want $|\frac{2n-2}{n+1} - 2| < \frac{1}{1000}$. We then perform the following algebra:

$$\begin{aligned}
\left|\frac{2n-2}{n+1} - 2\right| &< \frac{1}{1000} \\
\left|\frac{2n-2-2(n+1)}{n+1}\right| &< \frac{1}{1000} && \text{find a common denominator} \\
\left|\frac{2n-2-2n-2}{n+1}\right| &< \frac{1}{1000} \\
\left|\frac{-4}{n+1}\right| &< \frac{1}{1000} \\
\frac{4}{n+1} &< \frac{1}{1000} && \text{take the absolute value} \\
4000 &< n+1 && \text{cross multiply} \\
n &> 3999
\end{aligned}$$

6. Consider a geometric sequence with $a_3 = 48$ and $a_7 = 3$. Find the formula for a_n and the limit. Geometric sequences have the form $a_n = a_1 r^{n-1}$. We then plug in our given values and so have $48 = a_1 r^{3-1} = a_1 r^2$ and $1 = a_1 r^{7-1} = a_1 r^6$. We can then divide these two equations and get $\frac{48}{3} = \frac{a_1 r^2}{a_1 r^6}$ and so $16 = \frac{1}{r^4}$. We then cross multiply and so

$$r^4 = \frac{1}{16} = \frac{1}{2^4} = \left(\frac{1}{2}\right)^4,$$

and thus $r = \frac{1}{2}$. We now plug this back into our first equation and we have $48 = a_1 \left(\frac{1}{2}\right)^2$ and so $a_1 = 192$ and $a_n = 192 \left(\frac{1}{2}\right)^{n-1}$. We can then use this to find the limit. Because we have a geometric sequence with $-1 < r < 1$ by our geometric sequence rules, $\lim_{n\to\infty} a_n = 0$.

7. Consider the finite arithmetic sequence

$$-26, -23, ..., 34.$$

Find the formula for a_n and the number of terms. From the first two terms, since we know we have a arithmetic sequence, we know the common difference is 3. Thus the equation for $a_n = -26 + 3(n-1)$ and we just need to determine what term is 34. Then $34 = -26 + 3(n-1)$ and so $60 = 3(n-1)$. Then $n - 1 = 20$ and $n = 21$. Thus our sequence has 21 terms.

8. Suppose $\lim_{n\to\infty} a_n = 2$, $\lim_{n\to\infty} b_n = 4$, $\lim_{n\to\infty} c_n = 0$. Using the Limit Laws, find the following limits if possible. If not possible, explain why.