EE 503 : Problem Set #7 Solutions

7-1 a) From the problem statement, we have

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & \text{if } 0 \leq y \leq x \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = \begin{cases} 2 & \text{if } 0 \leq y \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The following graph is the sketch of the pdf where the shaded area has a value of 2 and the unshaded area has 0.

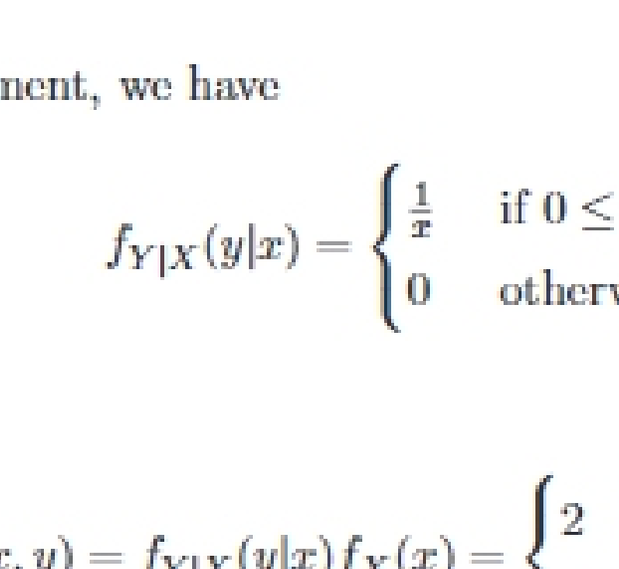

Figure 1: 7-1a

b) Since Joe will not win any money, the corresponding probability is given by

$$P(Y < 0.5 | X < 0.5) = \frac{P(Y < 0.5, X < 0.5)}{P(X < 0.5)} = \frac{P(X < 0.5)}{P(X < 0.5)} = 1.$$

c) We have

$$\begin{aligned} f_Y(y) &= \int_0^1 f_{X,Y}(x,y) \mathrm{d}x \\ &= \begin{cases} \int_y^1 2 \mathrm{d}x & 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 2(1-y) & 0 \leq y < 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The graph of the pdf is given in Figure 2.

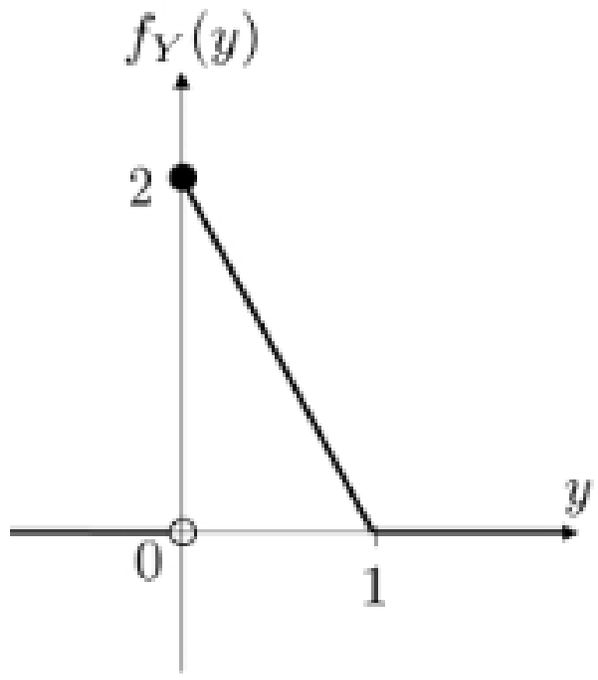

Figure 2: 7-1c

d) Let $W = X$. So $X = W$ and $Y = W - Z$. Then the Jacobian is given by

$$J = \begin{vmatrix} \frac{dZ}{dX} & \frac{dZ}{dY} \\ \frac{dW}{dX} & \frac{dW}{dY} \end{vmatrix} = 1.$$

Then

$$\begin{aligned} f_{Z,W}(z,w) &= f_{X,Y}(w, w-z)\frac{1}{|J|} \\ &= \begin{cases} 2 & \text{if } 0 \le w - z < w < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 2 & \text{if } 0 < z \le w < 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So

$$\begin{aligned} f_Z(z) &= \int_0^1 f_{Z,W}(z,w)\mathrm{d}w \\ &= \begin{cases} \int_z^1 2\mathrm{d}w & \text{if } 0 < z < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 2(1-z) & \text{if } 0 < z < 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

And $E[Z] = \int_0^1 2(z - z^2)\mathrm{d}z = \frac{1}{3}$ The graph of the pdf is given in Figure 3.

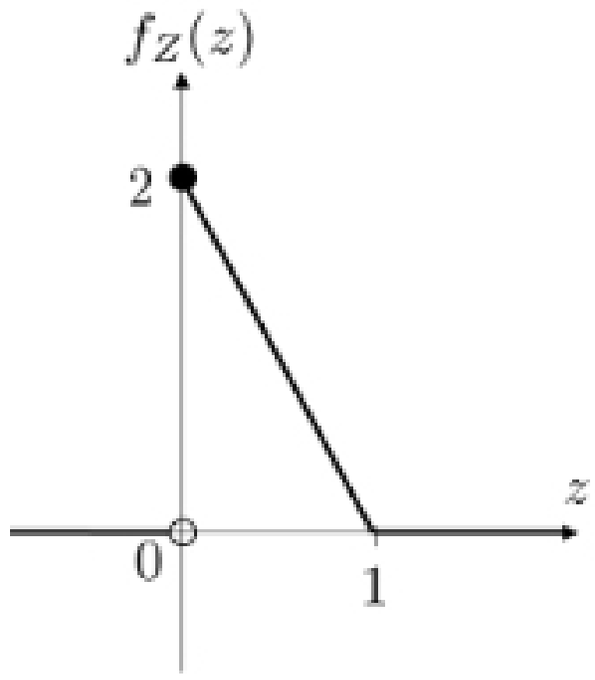

Figure 3: 7-1d

e) For the first event, the corresponding probability is given by

$$
\begin{aligned}
P(X < 0.5 | Y < 0.25) &= \frac{P(Y < 0.25, X < 0.5)}{P(Y < 0.25)} \\
&= \frac{\int_0^{0.5} \int_0^{0.25} f_{X,Y}(x,y) \mathrm{d}y \mathrm{d}x}{\int_0^{0.25} f_Y(y) \mathrm{d}y} \\
&= \frac{\int_0^{0.25} \int_0^{0.25} f_{X,Y}(x,y) \mathrm{d}y \mathrm{d}x + \int_{0.25}^{0.5} \int_0^{0.25} f_{X,Y}(x,y) \mathrm{d}y \mathrm{d}x}{\int_0^{0.25} 2(1-y) \mathrm{d}y} \\
&= \frac{\int_0^{0.25} \int_0^{x} 2 \mathrm{d}y \mathrm{d}x + \int_{0.25}^{0.5} \int_0^{0.25} 2 \mathrm{d}y \mathrm{d}x}{\int_0^{0.25} 2(1-y) \mathrm{d}y} \\
&= \frac{\frac{3}{16}}{\frac{7}{16}} \\
&= \frac{3}{7}.
\end{aligned}
$$