Worksheet 2. Math 231 AL1, BL1, DL1. Spring 2015. 1/23/2015.

Solutions

1. Evaluate using integration by parts.

(a) $\displaystyle\int \arctan x\, dx$

Solution: Set

$$\begin{aligned} u &= \arctan x & dv &= dx \\ du &= \frac{1}{1+x^2}\,dx & v &= x \end{aligned}$$

So using integration by parts gives us:

$$\int \arctan x\, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx$$

To deal with this new integral, a substitution will suffice. Set

$$\begin{aligned} w &= 1+x^2 \\ dw &= 2x\, dx \end{aligned}$$

so

$$\int \frac{x}{1+x^2}\, dx = \frac{1}{2}\int \frac{1}{w}\, dw = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln|1+x^2| + C.$$

(Note that $1+x^2$ is positive for all real x, so we can drop the absolute value bars.) Therefore,

$$\int \arctan x\, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx = \boxed{x\arctan x - \frac{1}{2}\ln(1+x^2) + C}$$

(b) $\displaystyle\int \frac{\ln x}{x^2}\, dx$

Solution: Set

$$\begin{aligned} u &= \ln x & dv &= x^{-2}\, dx \\ du &= x^{-1}\, dx & v &= -x^{-1} \end{aligned}$$

Using integration by parts, we get:

$$\int \frac{\ln x}{x^2}\, dx = -x^{-1}\ln x + \int x^{-2}\, dx = \boxed{-x^{-1}\ln x - x^{-1} + C}$$

(c) $\displaystyle\int t^3 e^{t^2}\, dt$ **(Hint: Substitute $x = t^2$)**

Solution: Start with the substitution

$$\begin{aligned} x &= t^2 \\ dx &= 2t\, dt \end{aligned}$$

to get

$$\int t^3 e^{t^2}\,dt = \int t^2 e^{t^2} t\,dt = \frac{1}{2}\int x e^x\,dx$$

Now we'll use integration by parts: set

$$\begin{array}{ll} u = x & dv = e^x\,dx \\ du = dx & v = e^x \end{array}$$

to obtain

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + C.$$

Therefore,

$$\int t^3 e^{t^2}\,dt = \frac{1}{2}\int x e^x\,dx = \frac{1}{2}(x e^x - e^x) + C = \boxed{\frac{1}{2}\left(t^2 e^{t^2} - e^{t^2}\right) + C}$$

2. (a) Integrate by parts to get a formula for $\int \sin^2 x\,dx$ which involves $\int \cos^2 x\,dx$.

Solution: Define $I = \int \sin^2 x\,dx$. Set

$$\begin{array}{ll} u = \sin x & dv = \sin x\,dx \\ du = \cos x\,dx & v = -\cos x \end{array}$$

so by integration by parts:

$$\boxed{I = -\sin x \cos x + \int \cos^2 x\,dx}$$

(b) Evaluate $\int \sin^2 x\,dx$ by using part (a) and the identity $\cos^2 x = 1 - \sin^2 x$.

Solution: By (a) and the given identity,

$$\begin{aligned} I &= -\sin x \cos x + \int \cos^2 x\,dx \\ &= -\sin x \cos x + \int (1 - \sin^2 x)\,dx \\ &= -\sin x \cos x + \int dx - \int \sin^2 x\,dx \\ &= -\sin x \cos x + x - I + C \end{aligned}$$

As our goal is to find I, solving the above for I gives us:

$$2I = x - \sin x \cos x + C$$

which implies

$$\boxed{I = \tfrac{1}{2}\left(x - \sin x \cos x\right) + C}$$

3. Evaluate the integrals.

(a) $\int e^{\sqrt{x}}\,dx$ **(Hint:** Substitute $t = \sqrt{x}$)

Begin with the substitution

$$t = \sqrt{x}$$
$$dt = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2t}\,dx \implies dx = 2t\,dt$$

to get

$$\int e^{\sqrt{x}}\,dx = \int e^t 2t\,dt = 2\int te^t\,dt$$

Now we use integration by parts: setting

$$u = t \qquad dv = e^t\,dt$$
$$du = dt \qquad v = e^t$$

gives

$$\int te^t\,dt = te^t - \int e^t dt = te^t - e^t + C = \sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}} + C$$

Hence,

$$\int e^{\sqrt{x}}\,dx = 2\int te^t\,dt = \boxed{2\left(\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}}\right) + C}$$

(b) $\int (\ln x)^2\,dx$

Set

$$u = (\ln x)^2 \qquad dv = dx$$
$$du = \frac{2\ln x}{x}\,dx \qquad v = x$$

so by integration by parts we have

$$\int (\ln x)^2\,dx = x(\ln x)^2 - \int \frac{2\ln x}{x} x\,dx = x(\ln x)^2 - 2\int \ln x\,dx$$

To evaluate this new integral, we'll use integration by parts again: set

$$U = \ln x \qquad dV = dx$$
$$dU = x^{-1}\,dx \qquad V = x$$

to obtain

$$\int \ln x\,dx = x\ln x - \int xx^{-1}\,dx = x\ln x - \int dx = x\ln x - x + C$$

Therefore,

$$\begin{aligned}\int (\ln x)^2\,dx &= x(\ln x)^2 - 2\int \ln x\,dx \\ &= x(\ln x)^2 - 2\left(x\ln x - x\right) + C \\ &= \boxed{x(\ln x)^2 - 2x\ln x + 2x + C}\end{aligned}$$