Worksheet 7 Solutions

① $f(x) = \frac{2+e^{-x}}{x}$ and $g(x) = \frac{1}{x}$ are continuous on $[1,\infty)$ and $f(x) > g(x) > 0$ for all $x \geq 1$.

Also, we know $\int_1^\infty \frac{1}{x}\,dx$ diverges.

By comparison test, $\int_1^\infty \frac{2+e^{-x}}{x}$ diverges.

② $\arctan x < \pi/2$ for all x and $2+e^x > e^x$ $\forall x$,

so $0 \leq \frac{\arctan x}{2+e^x} < \frac{\pi/2}{e^x}$ for $x \geq 0$.

Also, $f(x) = \frac{\arctan x}{2+e^x}$ and $g(x) = \frac{\pi/2}{e^x}$ are continuous on $[0,\infty)$. $\int_0^\infty \frac{\pi/2}{e^x}\,dx = \frac{\pi}{2}\int_0^\infty e^{-x}\,dx$ is known to converge, so $\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$ converges, by the comparison test.

③ $f(x) = \frac{\sin^2 x}{\sqrt{x}}$ and $g(x) = \frac{1}{\sqrt{x}}$ are continuous on $(0,\pi]$ and $0 \leq \frac{\sin^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ on $(0,\pi]$. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is known to converge,

so $\int_0^1 \frac{\sin^2 x}{\sqrt{x}}\,dx$ converges, by the comparison test.

4. ~~If $p=0$, then we have $\int_e^\infty \frac{1}{x}\,dx$,~~ ← Can't omit
which diverges.

If $\underline{p=1}$, we have $\displaystyle\int_e^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x\ln x}\,dx$

$\displaystyle = \lim_{t\to\infty} \ln(\ln t) - \ln(\ln e) = \infty$, so

diverges.

If $p \neq 1$, then $\displaystyle\int \frac{1}{x(\ln x)^p}\,dx = \int \frac{1}{u^p}\,dp = \int u^{-p}\,dp$

$\displaystyle = \frac{1}{1-p}u^{1-p} = \frac{1}{1-p}(\ln x)^{1-p}$. So

$$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}(\ln x)^{1-p}\Big|_e^t$$

$$= \lim_{t\to\infty} \frac{1}{1-p}(\ln t)^{1-p} - \frac{1}{1-p}\cdot 1 = \frac{1}{p-1} \quad \text{if } 1-p<0,\ \underline{p>1}$$

diverges if $1-p>0$; $\underline{p<1}$

5. $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx = \underbrace{\int_0^1 \frac{1}{x^p}\,dx}_{\text{diverges for } p\geq 1} + \underbrace{\int_1^\infty \frac{1}{x^p}\,dx}_{\text{diverges for } p\leq 1}$

Therefore $\int_0^\infty \frac{1}{x^p}\,dx$ diverges for all p.

⑥ $$\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx$$

$$= \lim_{t\to-\infty}\int_{t}^{0} e^{x}\,dx + \lim_{t\to\infty}\int_{0}^{t} e^{-x}\,dx$$

$$= \lim_{t\to-\infty}\left(e^{0}-e^{t}\right) + \lim_{t\to\infty}\left(-e^{-t}+e^{0}\right)$$

$= 1+1=2$ Converges to 2.

⑦. $f(x)=x$. $\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$

both ~~converges~~ diverge

But $$\lim_{t\to\infty}\int_{-t}^{t} x\,dx = \lim_{t\to\infty} \tfrac{1}{2}x^{1/2}\Big|_{-t}^{t} = \lim_{t\to\infty} \tfrac{1}{2}\sqrt{t}-\tfrac{1}{2}\sqrt{t}$$

$$=0.$$