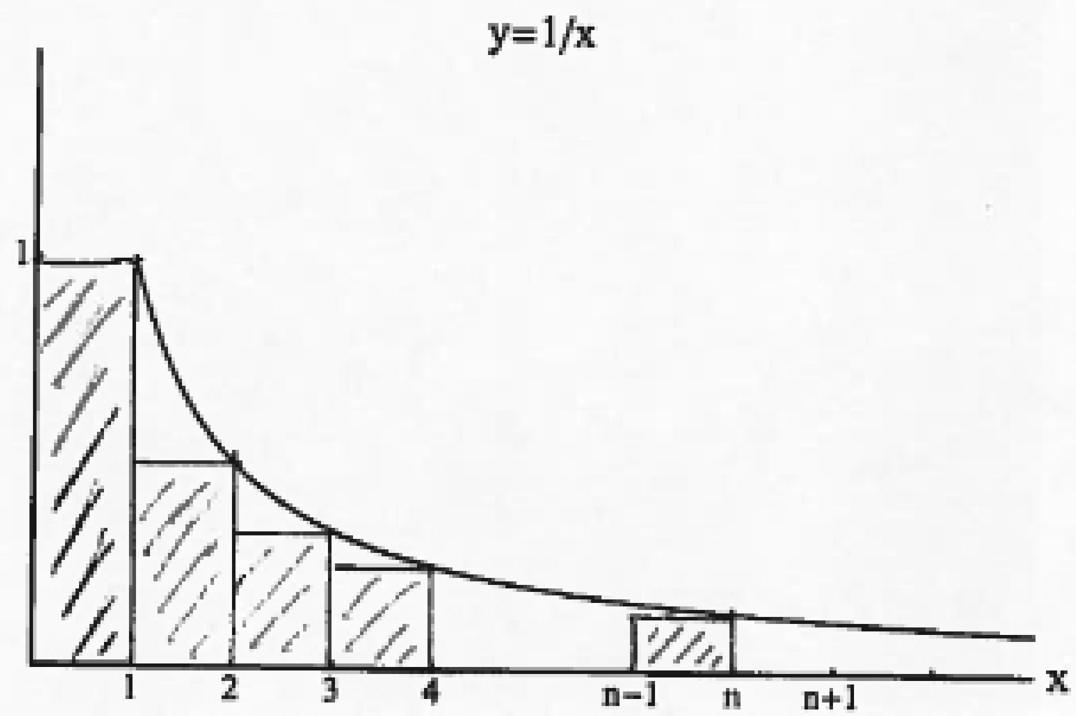
Group: ____________ ____________ ____________ Name: ____________

Math 231 Worksheet 13. March 6, 2015

1. Recall that the harmonic series is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$. Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ be the nth partial sum.

a) What is $\lim_{n\to\infty} s_n$? $= \infty$ (harmonic series diverges - from lecture).



b) Draw a careful picture on the graph to the right which illustrates that $s_n \le 1 + \ln n$. Be sure that your reasoning is explained.

$s_n =$ sum of areas of rectangles $\le 1 + \int_1^n \frac{1}{x}\,dx = 1 + \ln n$

1st rectangle.

c) Suppose that you were to add 3,000,000,000 terms of the harmonic series. Show that the sum would be less than 23.

$3{,}000{,}000{,}000 = 3 \times 10^9$

$S_{3\times 10^9} \le 1 + \ln(3\times 10^9) \approx 22.82 < 23$

and $f'(x) = \frac{-1}{x^2}\left[\frac{p + \ln x}{(\ln x)^{p+1}}\right] < 0$ for $x \ge 2$

so f is decreasing on $[2,\infty)$

2. Use the integral test to show that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$.

Let $f(x) = \frac{1}{x(\ln x)^p}$. On $[2,\infty)$, f is positive and continuous and $\lim_{x\to\infty} f(x) = 0$ for $p > 1$. Using the substitution $u = \ln x$, $\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \int_{\ln 2}^\infty \frac{1}{u^p}\,du = \frac{1}{p-1}(\ln 2)^{1-p}$ for $p > 1$.

By integral test, the series converges.

Recall that if integral test proves that a series converges, then

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx \qquad \text{and} \qquad S_n + \int_{n+1}^{\infty} f(x)\,dx < S < S_n + \int_{n}^{\infty} f(x)\,dx$$

where $a_n = f(n)$, and

$$S = \sum_{n=1}^{\infty} a_n, \qquad S_n = a_1 + a_2 + \cdots + a_n, \qquad R_n = a_{n+1} + a_{n+2} + \ldots$$

3. How many terms of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ would you need to add to estimate the sum to within 0.01? (Hint: you want $R_n < 0.01$.)

$$R_n < \int_n^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln n}^{\infty} \frac{1}{u^2}\,du = \lim_{t\to\infty} (-u^{-1})\Big|_{\ln n}^{t} = (\ln n)^{-1}$$

$\underline{\text{Want:}}$ $\frac{1}{\ln n} < 0.01 = \frac{1}{100} \Rightarrow \ln n > 100 \Rightarrow n > e^{100} \approx 2.69 \times 10^{43}$

4. a) Estimate the maximum possible error when the 20th partial sum $\sum_{n=1}^{20} \frac{1}{n^3}$ is used to estimate the sum $s = \sum_{n=1}^{\infty} \frac{1}{n^3}$.

$$R_{20} < \int_{20}^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \frac{x^{-2}}{-2}\Big|_{20}^{t} = \frac{1}{2\cdot 20^2} = \frac{1}{800}$$

b) The 20th partial sum is $s_{20} \approx 1.200867842...$ Find a short interval (a, b) which contains s.

$$\left.\begin{array}{l} \underbrace{s_{20} + \int_{21}^{\infty} \frac{1}{x^3}\,dx}_{s_{20} + \frac{1}{2\cdot 21^2} \approx 1.2020016} < s < \underbrace{s_{20} + \int_{20}^{\infty} \frac{1}{x^3}\,dx}_{s_{20} + \frac{1}{2\cdot 20^2} \approx 1.202117} \end{array}\right] \Rightarrow s \text{ is in } (1.2020016, 1.202117)$$