Group: ________________ Name: ______________________________

Math 231. Worksheet 14. March 11, 2015

1. Use Comparison or Limit Comparison to show convergence or divergence. State your choices of a_n and b_n, on each problem.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$

Comparison: $0 < \underbrace{\frac{1}{n^2+n+1}}_{a_n} < \underbrace{\frac{1}{n^2}}_{b_n}$ for all n

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converges.

Limit comparison: Same a_n, b_n. $\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1 > 0$.

$\sum \frac{1}{n^2}$ converges, so $\sum \frac{1}{n^2+n+1}$ converges.

(b) $\displaystyle\sum_{n=6}^{\infty} \frac{\sqrt{n^3+2}}{3n^2+14n+2}$ Limit comparison.

$a_n = \frac{\sqrt{n^3+2}}{3n^2+14n+2}$. $b_n = \frac{n^{3/2}}{n^2} = \frac{1}{n^{1/2}}$. $a_n > 0$, $b_n > 0$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{1/2}\sqrt{n^3+2}}{3n^2+14n+2} = \lim_{n\to\infty} \frac{n^2\sqrt{1+2/n^4}}{n^2(3+14/n+2/n^2)} = \frac{1}{3} > 0$

$\sum \frac{1}{n^{1/2}}$ diverges (p-series, $p \le 1$), so $\sum \frac{\sqrt{n^3+2}}{3n^2+14n+2}$ diverges

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n+4}$

Comparison $0 < \frac{2^n}{3^n+4} < \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$

$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ is a geometric series with $|r|<1$ ($r = 2/3$),

so it converges. By the comparison test, $\sum_{n=0}^{\infty} \frac{2^n}{3^n+4}$ converges

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n-n^2}$ for $n \ge 3$, $\frac{2^n}{3^n-n^2} > 0$

Limit Comparison. $a_n = \frac{2^n}{3^n-n^2}$ $b_n = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n \cdot 3^n}{(3^n-n^2)2^n} = \lim_{n\to\infty} \frac{1}{1-n^2/3^n} = 1 > 0$

By the limit comparison test, $\sum \frac{2^n}{3^n-n^2}$ converges because $\sum b_n = \sum \left(\frac{2}{3}\right)^n$ is a convergent geometric series.

2. (Showing why we want the limit to be nonzero, in the Limit Comparison Test)

Find an example of series $\sum a_n$ and $\sum b_n$ with **positive terms** that satisfy $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, $\sum a_n$ convergent, $\sum b_n$ divergent. (*Note.* The Limit Comparison does *not* apply to this example, because the limit of a_n/b_n equals 0.)

A simple example is $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n}$.

$\sum \frac{1}{n^2}$ converges, $\sum \frac{1}{n}$ diverges,

and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^2}{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0$.

An even simpler example is $b_n = 1$ for all n and $\sum a_n$ = any ~~positive~~ convergent sequence with positive terms.

3. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges or diverges. (Hint: compare with a geometric series.)

$$n! = \underbrace{1\cdot 2\cdot 3\cdot \cdots \cdot n}_{n \text{ factors}} \geq \underbrace{2\cdot 2\cdot \cdots \cdot 2}_{n-1 \text{ factors}} = 2^{n-1} \quad \text{for } n \geq 2$$

n-1 factors

Therefore $0 < \frac{1}{n!} < \frac{1}{2^{n-1}}$ for $n \geq 2$.

$\sum_{n=0}^{\infty} \frac{1}{2^{n-1}}$ is a geometric series with $r = \frac{1}{2}$,

so it converges.

By the comparison test, $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.