Group: ______________ ______________ ______________ Name: ______________

Math 231 Worksheet 15. March 13, 2015

1. Consider the curve $y = e^{2x}$ between the points $(\frac{1}{2}, e)$ and $(1, e^2)$.

Set up **but do not evaluate** integrals which represent the following quantities.

a) The length of the curve. $dy/dx = 2e^{2x}$

$$\text{length} = \int_{\frac{1}{2}}^{1} \sqrt{1+4e^{4x}}\,dx$$

b) The surface area when the curve is rotated about the x-axis. **Your answer must be an integral with respect to x.**

$$\text{surface area} = \int_{\frac{1}{2}}^{1} 2\pi e^{2x}\sqrt{1+4e^{4x}}\,dx$$

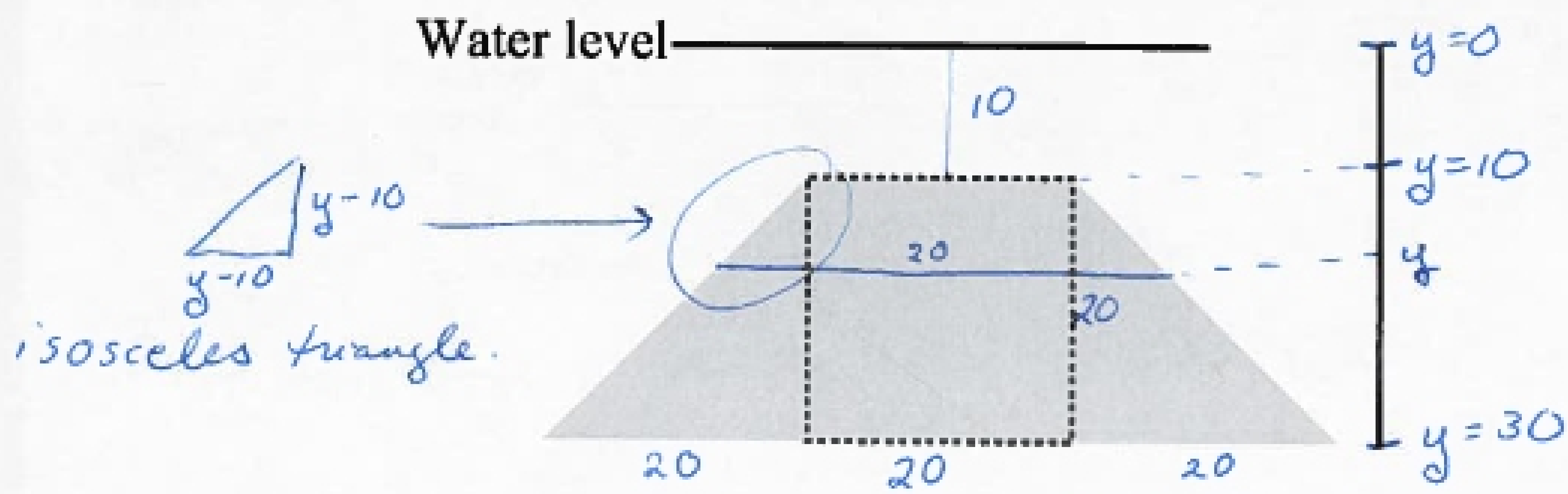
c) Give another integral which represents the surface area when the curve is rotated about the x-axis. **Your answer must be an integral with respect to y.**

$2x = \ln y$; $x = \frac{1}{2}\ln y$; $dx/dy = \frac{1}{2y}$

$$\text{surface area} = \int_{e}^{e^2} 2\pi y\sqrt{1+\frac{1}{4y^2}}\,dy$$

2. A vertical trapezoidal wall is shown. The dotted lines form a square of side length 20 m and the base is 60 m long. The top is 10 m under the water. Set up **but do not evaluate** an integral which represents the hydrostatic force on the dam. You must label your coordinates clearly on the vertical axis to the right.

Use ρ kg/m^3 for the density of water and g kg m/s^2 for the acceleration due to gravity.



area of slice at depth y is $[(y-10)+20+(y-10)]\,dy = 2y\,dy$

$$\text{Force} = \int_{10}^{30} \rho g\, \underset{\substack{\uparrow \\ \text{depth}}}{y}\, (2y\,dy) = 2\rho g \int_{10}^{30} y^2\,dy$$

3. Determine whether the sequence $\{a_n\}$ converges. If it converges, find the limit.

$a_n = \dfrac{\sin(\ln(n))}{e+\sqrt{n}}$

Since $-1 \le \sin(\ln n) \le 1$ for all n,

$$\frac{-1}{e+\sqrt{n}} \le \frac{\sin(\ln(n))}{e+\sqrt{n}} \le \frac{1}{e+\sqrt{n}} \text{ for all } n.$$

As $n \to \infty$, the limit of both $\frac{-1}{e+\sqrt{n}}$ and $\frac{1}{e+\sqrt{n}}$ is 0.

By the squeeze theorem, $\displaystyle\lim_{n\to\infty} \frac{\sin(\ln(n))}{e+\sqrt{n}} = 0$ also.

4. Determine whether the following series converges. If it converges, find its sum.

$\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot e^n}{\pi^{n-1}}$

This is a geometric series with $r = \frac{e}{\pi}$

since $|r|<1$, it converges.

The first term is $a = \frac{2e}{\pi^0} = 2e$.

It converges to $\dfrac{a}{1-r} = \dfrac{2e}{1-\frac{e}{\pi}}$

5. Determine whether the following series converges:

$\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

Integral test: $f(x) = \frac{\ln x}{x^2}$ is positive for $x>1$, continuous for $x>1$ and decreasing because $f'(x) = \frac{1-2\ln x}{x^3} < 0$ on $[2,\infty)$.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[\frac{-1}{x}\ln x - \frac{1}{x}\right]\Bigg|_1^t$$

(use integration by parts) $= \displaystyle\lim_{t\to\infty}\left[\frac{-1}{t}\ln t - \frac{1}{t} - (0-1)\right] = 1$

By the Integral test, $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n)}{n^2}$ converges.

6. Determine whether the following series converges:

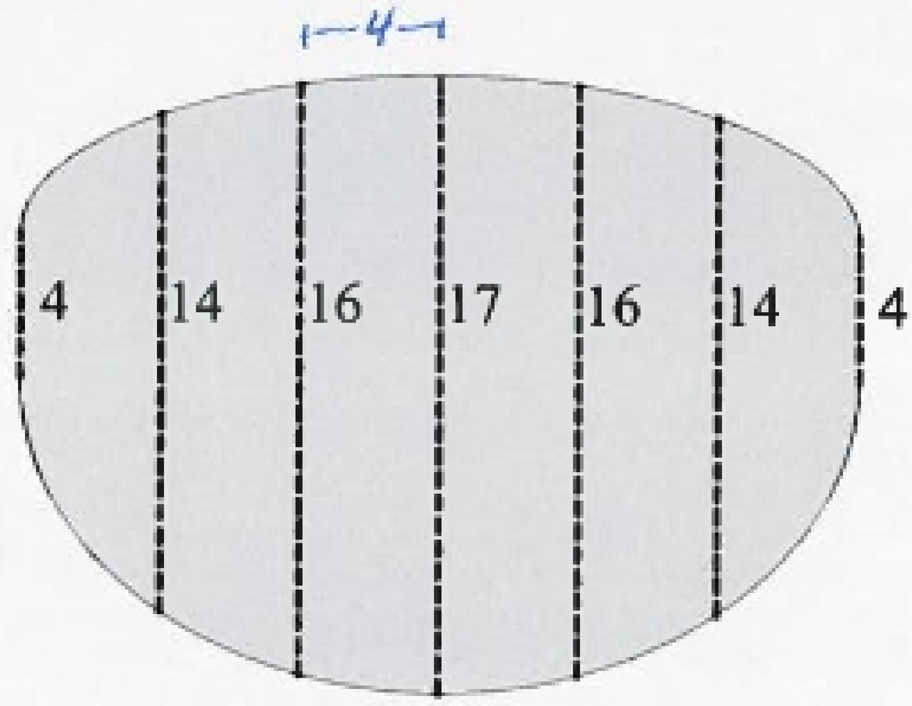
$\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{5n^4+3}} = \sum a_n$ Limit Comparison with $b_n = \dfrac{1}{n^{4/3}}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n^{4/3}}{\sqrt[3]{5n^4+3}} = \lim_{n\to\infty}\frac{2}{\sqrt[3]{5+3/n^4}} = \frac{2}{\sqrt[3]{5}} \leftarrow \text{positive and finite}$$

$\sum \frac{1}{n^{4/3}}$ conv. (p-series; $p>1$) so $\sum \frac{2}{\sqrt[3]{5n^4+3}}$ converges also.

Here are some extra practice problems. The problems on this third page will not be graded.

7. The cross-section of a small lake is measured at 4-meter intervals. The length of each cross-section (in meters) is given below.

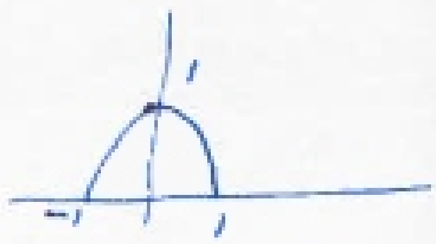


Estimate the area of the pond using Simpson's rule. **Do not simplify your answer.**

$\frac{4}{3}\left[4+4(14)+2(16)+4(17)+2(16)+4(14)+4\right]$

8. Let R be the region bounded by the curves

$y = 1 - x^2$ and $y = 0$.



a) Sketch the region R and find its area.

area $= \int_{-1}^{1} 1-x^2\,dx = \left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{1} = 1 - \frac{1}{3} - (-1+\frac{1}{3}) = 4/3$

b) Find the centroid of R. *Hint.* Finding $\bar{x}$ is easy (why?).

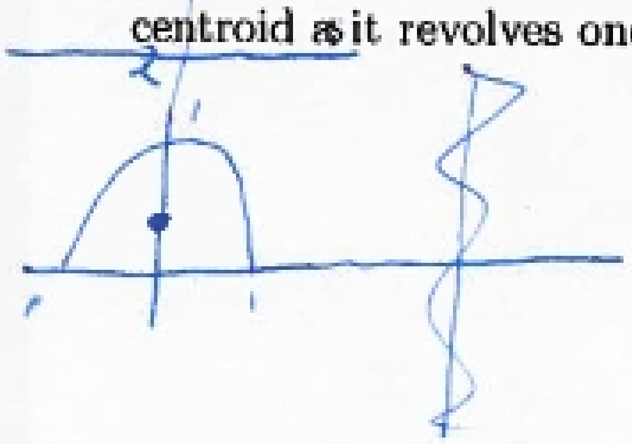
$\bar{x} = 0$ by symmetry.

$\bar{y} = \frac{1}{\text{area}} \int_{-1}^{1} \frac{1}{2}(1-x^2)^2\,dx = \frac{3}{8}\int_{-1}^{1} 1-2x^2+x^4\,dx = \frac{3}{8}\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_{-1}^{1}$

$= 2/5$

centroid $(\bar{x}, \bar{y}) = (0, 2/5)$

c) Find the volume of the solid formed by revolving R about the line $y = 2$. (Remember the Theorem of Pappus on page 559: volume = the are of R times the distance traveled by the centroid as it revolves once around the axis.



The centroid is $2 - \frac{2}{5} = \frac{8}{5}$ away from the line $y=2$. It travels $2\pi \cdot \frac{8}{5} = \frac{16\pi}{5}$ in one revolution.

Volume $= \frac{4}{3} \cdot \frac{16\pi}{5} = \frac{64\pi}{15}$