Chapter 20
Expected Value

We have been studying how to find the probability of a specific event. When we buy a raffle ticket, we are usually curious what the probability of a win is, but we are also interested in determining *how much* we will win, on average.

Example 1

Suppose a sorority is selling 1000 raffle tickets for $1.00 each. One ticket will be drawn at random and the winner receives $200.00. There are two possible outcomes here:

1. Win $200.00 (so net gain is $199 with probability 1/1000)
2. Do not win (so net "gain" is -$1 with probability 999/1000)

So how would we find the average gain?

Definition

The **expected value** of a random phenomenon that has numerical outcomes is found by multiplying each outcome by its probability and then adding all the products.

In symbols, if the possible outcomes are $a_1, a_2, a_3, \ldots, a_k$, and their probabilities are $p_1, p_2, p_3, \ldots, p_k$, then

$$\text{expected value} = a_1p_1 + a_2p_2 + a_3p_3 + \ldots + a_kp_k$$

Notation

We call E[ ] the expected value operator. We read "E[gain]" as "expected gain".

Note

An expected value is an average of all the possible outcomes.

Example 1 (cont'd…)

For the sorority fundraiser example find the expected gain (noticing the use of the definition of expected value this time).

Example 2
Consider a lottery game where you pay $0.50 and choose a three digit number. The lottery picks a three digit number randomly and pays you $250 if your number matches.
What is the probability model for your gain?

What is your long-run average gain? That is, find your expected gain.

Example 3
An insurance company sells a particular policy that costs the insured $1,000 to purchase. The insurance company knows that 1/100 of these policy holders will file a claim worth $20,000, 1/200 will file a $50,000 claim, and 1/500 will file a claim of $100,000. Find the insurance company's expected gain for this policy.

Clicker Example
Suppose we have the following distribution of outcomes for a "loaded" die.

Outcome	1	2	3	4	5	6
Probability	0.5	0.1	0.1	0.1	0.1	0.1

Find the expected number you'll get when rolling this die.
A. 1 B. 1.5 C. 2 D. 2.5 E. 3

Example 4
Use simulation to estimate the expected number of children a couple will have if they follow the scheme that they will have children until they have a girl or until they have three children. Recall, the probability of a girl is 0.49 and a boy is 0.51.

Thinking about Expected Value as a Long-run Average

Notice that the expected value of a variable is a weighted average of all the possible outcomes. And it represents the long-run average we will actually see if we observe the random phenomenon many times. It makes sense intuitively to believe that the more trials we observe, the closer our observed average should be to the true average. This can also be proved mathematically and this result is known as the Law of Large Numbers.

> **Law of Large Numbers** If a random phenomenon with numerical outcomes is repeated many times independently, the mean of the observed outcomes approaches the expected value.

- The Law of Large Numbers is closely related to the idea of probability – in many independent repetitions of a random outcome, the proportion of times a random outcome occurs approaches the probability of that random outcome and the average outcome obtained will approach the expected value (as you increase the number of observations)
- The Law of Large Numbers explains why gambling is a business for a casino – they use the fact that the average winnings of a large number of customers will be quite close to its expected value and that is how they make money
- The more variable a random outcome is, the more trials are necessary for the mean of the observations to be close to the expected value