Name: Solutions

ID#: ____________

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Problem	Weight	Score
1	25	
2	25	
3	25	
4	25	
Total	100	

This test consists of four problems. Answer each problem on the exam itself; if you use additional paper, repeat the identifying information above, and staple it to the rest of your exam when you hand it in. The quality of your analysis and evaluation is as important as your answers. Your reasoning must be precise and clear; your complete English sentences should convey what you are doing.

Problem 1: (25 points)

A SISO LTI continuous-time system with output $y(t)$ and input $u(t)$ is described by the transfer function

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s^2+1}{s^3+2s+1}$$

1. (5 points) Find an ODE description of the system.

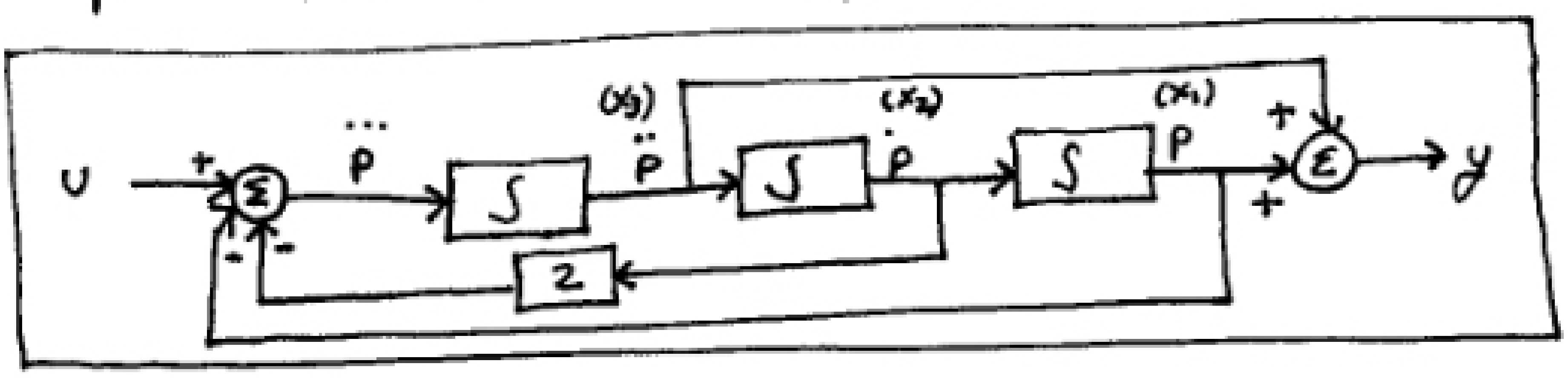
$$s^3 Y(s) + 2s\,Y(s) + Y(s) = s^2 U(s) + u(s)$$

$$\boxed{\dddot{y} + 2\dot{y} + y = \ddot{u} + u}$$

2. (12 points) Represent the system by an all-integrator block diagram that uses the smallest number of integrators.

$$\frac{Y(s)}{U(s)} = \frac{P(s)}{U(s)}\frac{Y(s)}{P(s)} = \left(\frac{1}{s^3+2s+1}\right)(s^2+1)$$

$$\dddot{p} = -2\dot{p} - p + u \qquad \text{and} \qquad y = \ddot{p} + p$$



Note that

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -x_1 - 2x_2$$

$$y = x_1 + x_3$$

3. (8 points) Find a state-space representation

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned}$$

for the system.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 0 \end{pmatrix}}_{F} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{G} u$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}}_{H} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Problem 2: (25 points)

Fluid flows play an important role in many physical systems. As an example, hydraulic fluid flows are used to move the control surfaces on airplanes. In this problem you will analyse the water tank system shown in Figure 1. Water enters at the top of the tank with a mass flow rate of $\omega_{\text{in}}(t)$ [Kg/sec] and exits at the bottom of the tank with a mass flow rate $\omega_{\text{out}}(t)$ [Kg/sec]. The fluid level $h(t)$ satisfies

$$\dot{h}(t) = \frac{1}{A_T \rho}\left[\omega_{\text{in}}(t) - \omega_{\text{out}}(t)\right],$$

where A_{T} [m²] is the cross-sectional area of the tank and ρ [Kg/m³] is the density of water. This continuity relation is simply a statement of the conservation of matter. If the input and output flow rates are equal, then the fluid level remains constant. On the other hand, if $\omega_{\text{in}} > \omega_{\text{out}}$, then the fluid level $h(t)$ increases. The fluid exits the tank through a discharge nozzle with cross-sectional area A_{N}. The output flow rate is given by Toricelli's equation

$$\omega_{\text{out}}(t) = \rho A_{\text{N}} \sqrt{\frac{2g}{1 - \frac{A_{\text{N}}^2}{A_{\text{T}}^2}}}\ \sqrt{h(t)},$$

where $g = 9.8$ m/sec² (use the symbol g in your analysis, do not substitute the numeric value for g).

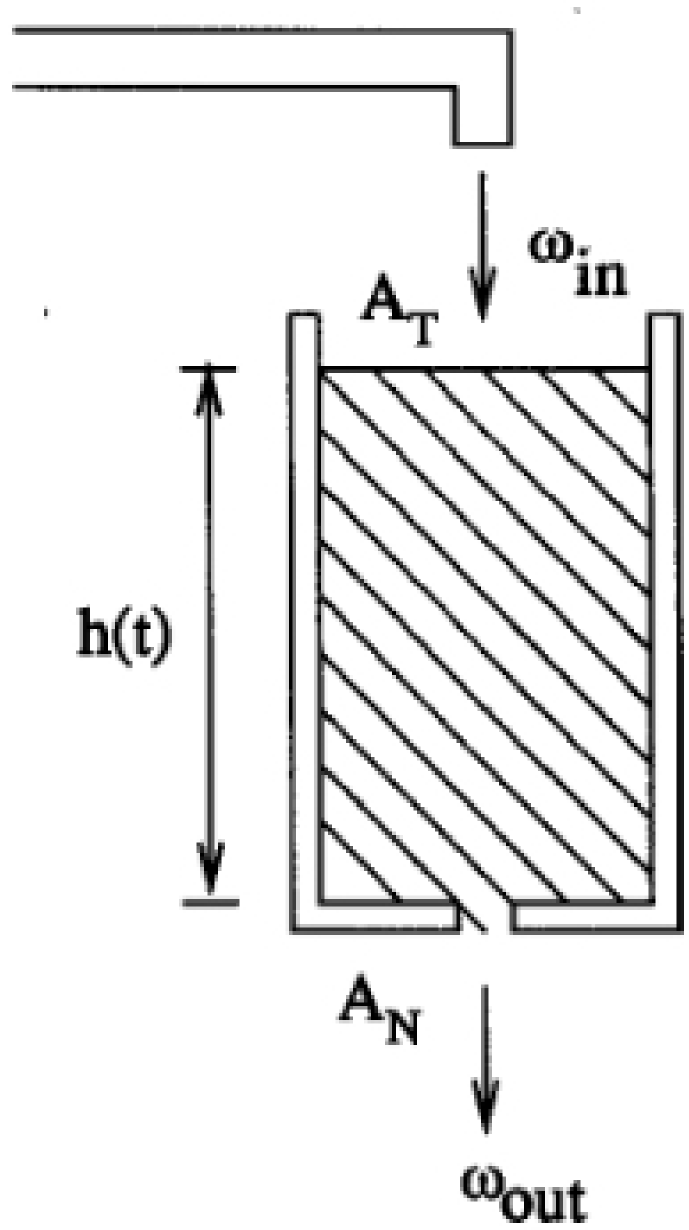


Figure 1: Elementary system for studying fluid flow dynamics.