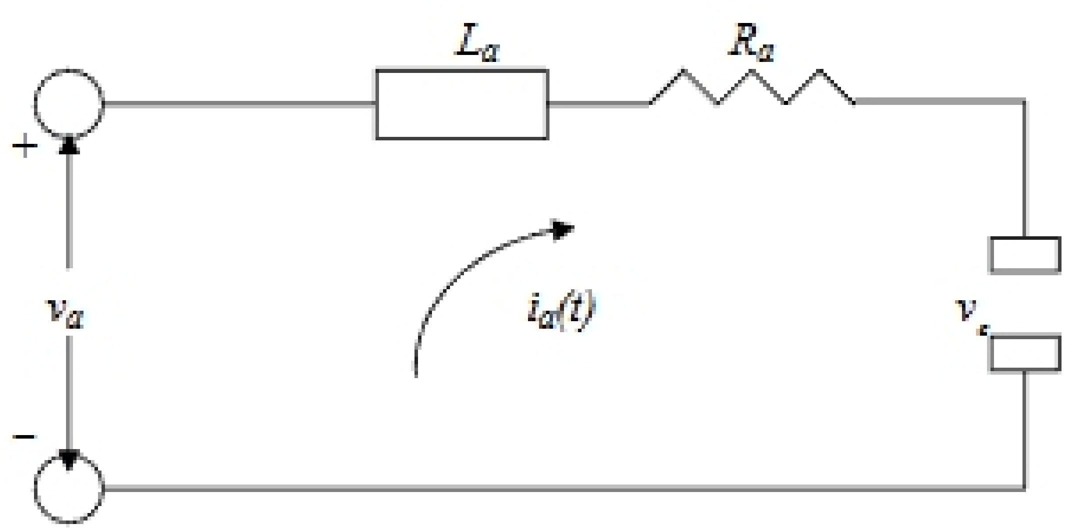
Solutions to Final Exam

1)

The motor is characterized by the following two equations:

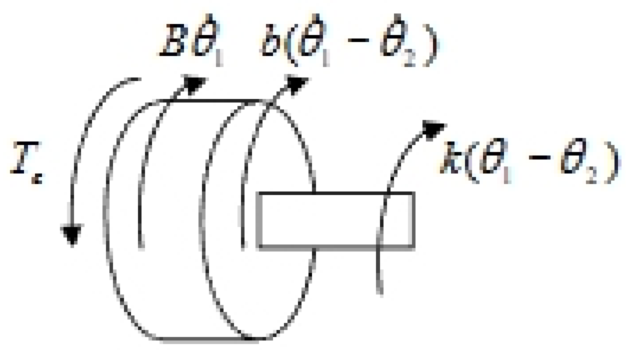
$$T_e = K_t i_a(t),$$
$$v_e = K_e \dot{\theta}$$

Writing the voltage balance equation for the electric circuit, we have



$$L_a \frac{di_a(t)}{dt} + i_a(t)R_a + K_e \dot{\theta}_1 = v_a. \quad (1)$$

Free body diagram of the electric motor



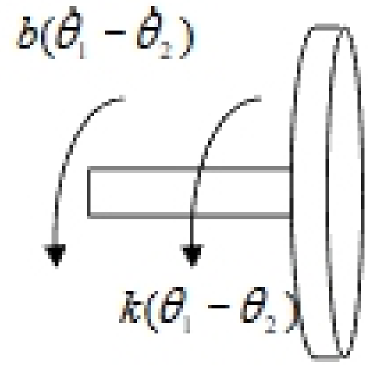
We know that, $J_1\ddot{\theta}_1 = T_e + \sum T$

$T_e = K_t i_a(t)$ and $\sum T = -B\dot{\theta}_1 - b(\dot{\theta}_1 - \dot{\theta}_2) - k(\theta_1 - \theta_2)$.

Substituting the values we get

$$J_1\ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) + B\dot{\theta}_1 = K_t i_a \quad (2)$$

Free body diagram of the shaft and load



Similarly, from the moment balance equation for shaft and load we have:

$$J_2\ddot\theta_2 = \sum T$$

and $\sum T = b(\dot\theta_1 - \dot\theta_2) + k(\theta_1 - \theta_2) \Rightarrow$

$$J_2\ddot\theta_2 + b(\dot\theta_2 - \dot\theta_1) + k(\theta_2 - \theta_1) = 0 \quad (3)$$

b)

Let $x_1 = \theta_2, x_2 = \dot\theta_2, x_3 = \theta_1, x_4 = \dot\theta_1, x_5 = i_a(t)$

Using the above relations and equations we get

$$\dot x_1 = x_2,$$
$$\dot x_2 = -\frac{k}{J_2}x_1 - \frac{b}{J_2}x_2 + \frac{k}{J_2}x_3 + \frac{b}{J_2}x_4$$
$$\dot x_3 = x_4$$
$$\dot x_4 = \frac{k}{J_1}x_1 + \frac{b}{J_1}x_2 - \frac{k}{J_1}x_3 - \frac{(b+B)}{J_1}x_4 + \frac{K_t}{J_1}x_5$$
$$\dot x_5 = -\frac{R_a}{L_a}x_5 - \frac{K_e}{L_a}x_4 + \frac{1}{L_a}v_a$$

Writing in matrix form we get,

$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\\ \dot x_5\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ -\frac{k}{J_2} & -\frac{b}{J_2} & \frac{k}{J_2} & \frac{b}{J_2} & 0\\ 0 & 0 & 0 & 1 & 0\\ \frac{k}{J_1} & \frac{b}{J_1} & -\frac{k}{J_1} & -\frac{(b+B)}{J_1} & \frac{K_t}{J_1}\\ 0 & 0 & 0 & -\frac{K_e}{L_a} & -\frac{R_a}{L_a}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\ \frac{1}{L_a}\end{bmatrix}v_a$$

Since the output of the system is defined to be the angular displacement of the load, we get

$Y = \theta_2 = x_1,$

$$\Rightarrow Y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

2) The nonlinear differential equation is given by

$$m\ddot{x}(t) + 2c(x^2 - 1)\dot{x}(t) + kx = 0.$$

To determine the equilibrium point, equate all the derivative terms to zero. Therefore we get

$x = 0$, is the equilibrium value of the variable, 'x'.

Define a new variable

$$\Delta x = x(t) - x_0,$$

where 'x_0' is the equilibrium value of, 'x'.

Applying Taylor series expansion to the above equation, we get

$$m\Delta\ddot{x}(t) + 2cx^2(t)\dot{x}(t)\Big|_{x=0} + \frac{\partial}{\partial x}\left(2cx^2(t)\dot{x}(t)\right)\Big|_{x=0} \cdot (x(t) - x_0) + \frac{\partial}{\partial \dot{x}}\left(2cx^2(t)\dot{x}(t)\right)\Big|_{x=0} \cdot (\dot{x}(t) - \dot{x}_0)$$

$$- 2c\Delta\dot{x}(t) + k(\Delta x(t) + x_0) = 0,$$

$$\Rightarrow m\Delta\ddot{x}(t) - 2c\Delta\dot{x}(t) + k\Delta x(t) = 0.$$