AAE 340 - PS10 S13 SOLUTION - DAVIDE GUZZETTI

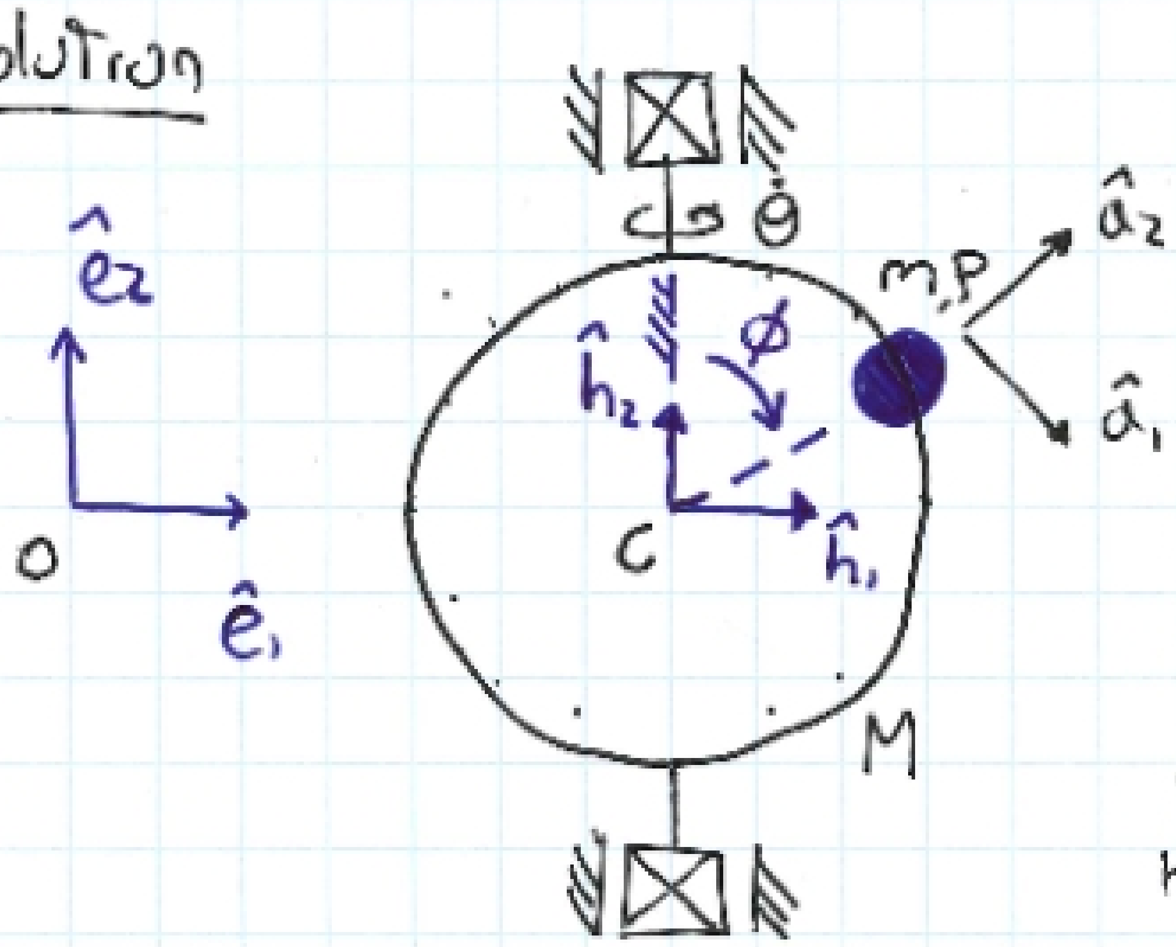
PROBLEM 1

Given
- A thin loop of mass M and radius R
- Hoop is FREE to rotate about vertical axis
- Particle of mass m can slide on the Frictionless hoop
- C located @ the center of loop

Find
(a) DOF's. Justify

(b) For the system: ${}^{e}\bar{H}^{c}$, $\dfrac{{}^{e}d\,{}^{e}\bar{H}^{c}}{dt}$, T_{TRANS}, T_{ROT}, T_{total}

Solution



$\hat{e}$: inertially Fixed
$\hat{e}_3 = \hat{e}_1 \times \hat{e}_2$

$\hat{h}$: Fixed in hoop
$\hat{h}_3 = \hat{h}_1 \times \hat{h}_2$

$\hat{a}$: Fixed in particle
$\hat{a}_3 = \hat{a}_1 \times \hat{a}_2$

$${}^{e}\bar{\omega}^{h} = \dot{\theta}\,\hat{h}_2 = \dot{\theta}\,\hat{e}_2$$
$${}^{h}\bar{\omega}^{a} = -\dot{\phi}\,\hat{h}_3 = -\dot{\phi}\,\hat{a}_3$$

(a) Loop = Rigid Body = max 6 DOFs

Trans DOF - max 3		Rot DOF - max 3	
Choose: pt C		Orient hoop about pt C	
* carthesian (3 distances)		Rotations Body 2-1-3	
$\bar{r}^{OC} = x\,\hat{e}_1 + y\,\hat{e}_2 + z\,\hat{e}_3$		θ about $\hat{h}_2$	
Coord: x, y, z	3	α about $\hat{h}_1$	3
		β about $\hat{h}_3$	
constraints: $x=x_0, y=y_0, z=z_0$	-3	constraints: $\alpha = 0$, $\beta = 0$	-2

TOTAL = 1 ROT DOF for the hoop

Particle - max 3DOF's

spherical (1 distance, 2 angle)
$\bar{r}^{CP} = r\,\hat{a}_2$
then, ψ and ϕ orient $\hat{a}$'s

coord: r, ψ, ϕ 3

Constraints $r =$ constant, $\psi = \theta$ -2

TOTAL = 1 DOF For the particle

* TOTAL DOF For the system $\Rightarrow$ 2DOF's

(b) Derive expression ${}^e\bar{H}^C$, ${}^e d\,{}^e\bar{H}^C/dt$

$${}^e\bar{H}^C = \left({}^e\bar{H}^C\right)_{hoop} + \left({}^e\bar{H}^C\right)_{particle}$$

$$= [I^C_{\hat{h}}]\{{}^e\bar{\omega}^h_{\hat{h}}\} + \bar{r}^{CP} \times m\,{}^e\bar{V}^P$$

where

$$[I^C_{\hat{h}}] = \begin{bmatrix} mR^2/2 & 0 & 0 \\ 0 & mR^2/2 & 0 \\ 0 & 0 & MR^2 \end{bmatrix} \leftarrow$$ From the TABLES!

$${}^e\bar{\omega}^h = \dot{\theta}\,\hat{h}_2$$

$$\bar{r}^{CP} = r\hat{a}_2 = Rc\phi\,\hat{h}_2 + Rs\phi\,\hat{h}_1$$
$${}^e\bar{V}^P = R\dot{\phi}c\phi\,\hat{h}_1 - R\dot{\phi}s\phi\,\hat{h}_2 - R\dot{\theta}s\phi\,\hat{h}_3 \quad \text{(BKE)}$$

$$\left({}^{e}\bar{H}^{c}_{\hat{h}}\right)_{hoop} = [I^{c}]_{\hat{h}}\left\{{}^{e}\bar{\omega}^{h}\right\}_{\hat{h}} = MR^2 \begin{bmatrix} 1/2 & & \emptyset \\ & 1/2 & \\ \emptyset & & 1 \end{bmatrix}_{\hat{h}} \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix}_{\hat{h}} = \begin{Bmatrix} 0 \\ \frac{1}{2}MR^2\dot{\theta} \\ 0 \end{Bmatrix}_{\hat{h}}$$

$$= \frac{1}{2} M R^2 \dot{\theta}\, \hat{h}_2$$

$$\left({}^{e}\bar{H}^{c}_{\hat{h}}\right)_{particle} = \bar{r}^{cp} \times m\,{}^{e}\bar{V}^{p} = -mR^2\dot{\theta} s_\phi c_\phi \hat{h}_1 + mR^2 s_\phi^2 \dot{\theta}\hat{h}_2 - mR^2\dot{\phi}\hat{h}_3$$

$$\boxed{{}^{e}\bar{H}^{c} = -mR^2\dot{\theta} s_\phi c_\phi \hat{h}_1 + \left(m s_\phi^2 + \frac{M}{2}\right)R^2\dot{\theta}\hat{h}_2 - mR^2\dot{\phi}\hat{h}_3} \quad *$$

$$\frac{{}^{e}d\,{}^{e}\bar{H}^{c}}{dt} = \frac{{}^{h}d\,{}^{e}\bar{H}^{c}}{dt} + {}^{e}\bar{\omega}^{h} \times {}^{e}\bar{H}^{c}$$

$$= -mR^2\left(\ddot{\theta} s_\phi c_\phi + \dot{\theta}\dot{\phi} c_\phi^2 - \dot{\theta}\dot{\phi} s_\phi^2\right)\hat{h}_1$$

$$+ \left(2m\dot{\phi} s_\phi c_\phi R^2 \dot{\theta} + \left(m s_\phi^2 + \frac{M}{2}\right) R \ddot{\theta}\right)\hat{h}_2$$

$$- mR^2\ddot{\phi}\hat{h}_3 - mR^2\dot{\theta}\dot{\phi}\hat{h}_1 + mR^2\dot{\theta}^2 c_\phi s_\phi \hat{h}_3$$

$$\frac{{}^{e}d\,{}^{e}\bar{H}^{c}}{dt} = -mR^2\left(\ddot{\theta} s_\phi c_\phi + c_{2\phi}\dot{\theta}\dot{\phi} + \dot{\theta}\dot{\phi}\right)\hat{h}_1$$

$$+ \left[mR^2\dot{\phi}\dot{\theta} s_{2\phi} + \left(m s_\phi^2 + M/2\right)R^2\ddot{\theta}\right]\hat{h}_2$$

$$+ mR^2\left(\dot{\theta}^2 c_\phi s_\phi - \ddot{\phi}\right)\hat{h}_3$$

$$* \quad \boxed{\frac{{}^{e}d\,{}^{e}\bar{H}^{c}}{dt} = -mR^2\left(\ddot{\theta} s_\phi c_\phi + 2\dot{\theta}\dot{\phi} c_\phi^2\right)\hat{h}_1 + \left[mR^2\dot{\phi}\dot{\theta} s_{2\phi} + \left(m s_\phi^2 + M/2\right)\cdot R^2\ddot{\theta}\right]\hat{h}_2 + mR^2\left(\dot{\theta}^2 c_\phi s_\phi - \ddot{\phi}\right)\hat{h}_3}$$