AAE 340 – Dynamics and Vibrations

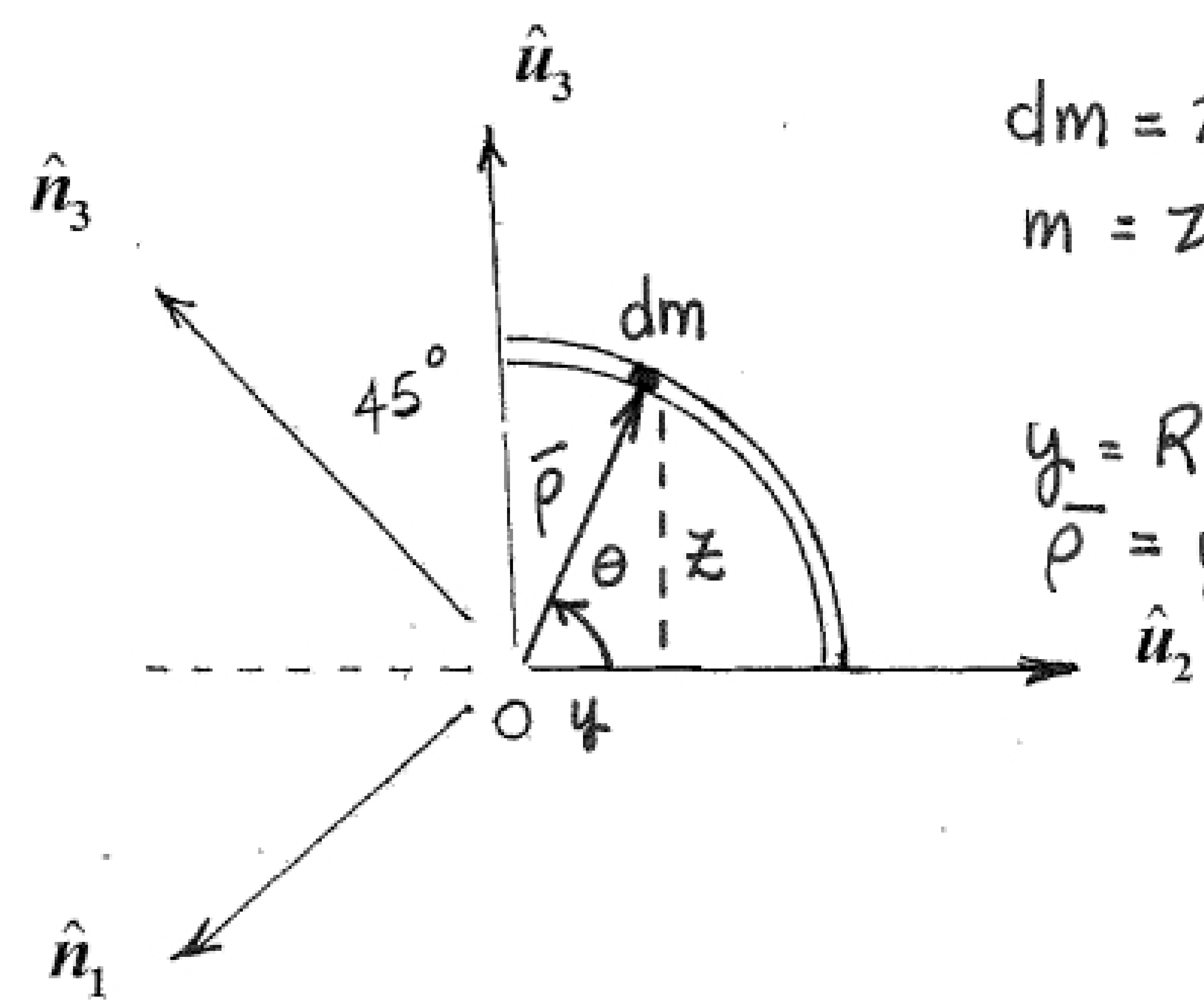
Exam III

Solution

Please read the problems carefully.
Write clearly and use diagrams when necessary.

(35 points)

1. Below is exactly one-quarter ring (constant density ν, total mass m, and radius R). Unit vectors $\hat{u}_i$ are fixed in the ring as indicated. Note that point O is located at the center of curvature of the ring.



$dm = \nu R\, d\theta$

$m = \nu R \dfrac{\pi}{2}$

$y = R c_\theta \qquad z = R s_\theta$

$\bar{\rho} = y\hat{u}_2 + z\hat{u}_3$

(a) Integrate and determine the following inertia element for point O associated with unit vectors $\hat{u}_i$, that is, I_{23}^O ; express it in terms of m and R.

(b) Assume that the complete inertia matrix is given as.

$$\left[I^O\right]_{\hat{u}} = \frac{mR^2}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{2}{\pi} \\ 0 & -\frac{2}{\pi} & 1 \end{bmatrix}$$

Determine the expression for $\left[I^O\right]_{\hat{n}}$.

(a) $I_{23}^{0} = -\int \rho_2 \rho_3 \, dm = -\int y z \nu R \, d\theta$

$$= -\int_0^{90^\circ} \nu (R C_\theta)(R S_\theta) R \, d\theta = -\nu R^3 \int_0^{90^\circ} C_\theta S_\theta \, d\theta$$

$$= -\nu \frac{R^3 S_\theta^2}{2} \Big]_0^{90} = -\frac{\nu R^3}{2} = -\left(\frac{\cancel{2}m}{R\pi}\right)\frac{R}{\cancel{2}} = -\frac{mR^2}{\pi}$$

$$\boxed{I_{23}^{0} = -\frac{mR^2}{\pi}}$$

(b) $$\underset{\hat{n}\cdot\hat{u}}{[\ell]} = \begin{bmatrix} 0 & -C45 & -S45 \\ 1 & 0 & 0 \\ 0 & -S45 & C45 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underset{\hat{n}}{[I^\circ]} = \underset{\hat{n}\cdot\hat{u}}{[\ell]} \, \underset{\hat{u}}{[I^\circ]} \, [\ell]^T$$

$$= \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \frac{mR^2}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{2}{\pi} \\ 0 & -\frac{2}{\pi} & 1 \end{bmatrix} \begin{bmatrix} 0 & & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{mR^2}{2} \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{2}{\pi} & \frac{1}{\sqrt{2}}\frac{2}{\pi} - \frac{1}{\sqrt{2}} \\ 2 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{}{\pi} & \frac{1}{\sqrt{2}}\frac{2}{\pi} + \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{mR^2}{2} \begin{bmatrix} \frac{1}{2} - \frac{1}{\pi} - \frac{1}{\pi} + \frac{1}{2} & 0 & \frac{1}{2} - \frac{1}{\pi} + \frac{1}{\pi} - \frac{1}{2} \\ 0 & 2 & 0 \\ \frac{1}{2} + \frac{1}{\pi} - \frac{1}{\pi} - \frac{1}{2} & 0 & \frac{1}{2} + \frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{2} \end{bmatrix}$$

$$\boxed{\underset{\hat{n}}{[I^\circ]} = \frac{mR^2}{2} \begin{bmatrix} 1 - \frac{2}{\pi} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 + \frac{2}{\pi} \end{bmatrix}}$$