

The Inverse of a Square Matrix

In this section we discuss a procedure for finding the inverse of a matrix and show how the inverse can be used to help us solve a system of linear equations.

Let A be a square matrix of size n . A square matrix A^{-1} of size n such that $AA^{-1} = A^{-1}A = I_n$ is called the inverse of A . $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note: Not every square matrix has an inverse. A matrix with no inverse is called singular.

Finding the Inverse of a Matrix

Given the $n \times n$ matrix A :

1. Adjoin the $n \times n$ identity matrix I to obtain the augmented matrix

$(A | I)$

2. Use a sequence of row operations to reduce $(A | I)$ to the form $(I | B)$, if possible.

The matrix B is the inverse of A .

Example 1: Find the inverse:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array} \right] \xrightarrow{R_1 + R_2 \rightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array} \right] \xrightarrow{\frac{1}{5}R_2 \rightarrow R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array} \right] \xrightarrow{-2R_2 + R_1 \rightarrow R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

Example 2: Find the inverse of a 3 x 3 matrix. (Use Gauss-Jordan)

$$\begin{aligned}
 C &= \begin{pmatrix} 1 & 4 & -1 \\ 2 & 3 & -2 \\ -1 & 2 & 3 \end{pmatrix} \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 3 & -2 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ R_1 + R_3 \rightarrow R_3 \end{array} \\
 & \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -5 & 0 & -2 & 1 & 0 \\ 0 & 6 & 2 & 1 & 0 & 1 \end{array} \right] R_2 + R_3 \rightarrow R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 6 & 2 & 1 & 0 & 1 \end{array} \right] \begin{array}{l} -4R_2 + R_1 \rightarrow R_1 \\ -6R_2 + R_3 \rightarrow R_3 \end{array} \\
 & \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 5 & -4 & -4 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & -10 & 7 & -6 & -5 \end{array} \right] R_3 \left(\frac{-1}{-10} \right) \rightarrow R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 5 & -4 & -4 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{7}{10} & \frac{3}{5} & \frac{1}{2} \end{array} \right] \begin{array}{l} 9R_3 + R_1 \rightarrow R_1 \\ -2R_3 + R_2 \rightarrow R_2 \end{array} \\
 & \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{13}{10} & \frac{7}{5} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{7}{10} & \frac{3}{5} & \frac{1}{2} \end{array} \right] A^{-1} = \begin{bmatrix} -\frac{13}{10} & \frac{7}{5} & \frac{1}{2} \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{7}{10} & \frac{3}{5} & \frac{1}{2} \end{bmatrix}
 \end{aligned}$$

Example 3: Find the inverse.

$$B = \begin{pmatrix} 4 & 2 & 2 \\ -1 & -3 & 4 \\ 3 & -1 & 6 \end{pmatrix} \left[\begin{array}{ccc|ccc} 4 & 2 & 2 & 1 & 0 & 0 \\ -1 & -3 & 4 & 0 & 1 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{array} \right] R_1 - R_3 \rightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & -1 \\ -1 & -3 & 4 & 0 & 1 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

No inverse, row of zeros on left and numbers on right

Matrices That Have No Inverses

If there is a row to the left of the vertical line in the augmented matrix containing all zeros, then the matrix does not have an inverse. Example 3 has this problem and does not have an inverse.

Formula for the Inverse of a 2X2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose $D = ad - bc$ is not equal to zero. Then, A^{-1} exists and is given by $A^{-1} = \frac{1}{D} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 4: Find the inverse of the following matrices.

a. $A = \begin{pmatrix} -5 & 10 \\ 2 & 7 \end{pmatrix}$ $d = -5(7) - 10(2) = -35 - 20 = -55$

$$A^{-1} = \frac{1}{-55} \begin{bmatrix} 7 & -10 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} -\frac{7}{55} & \frac{10}{55} \\ \frac{2}{55} & -\frac{5}{55} \end{bmatrix} = \begin{bmatrix} -\frac{7}{55} & \frac{2}{11} \\ \frac{2}{55} & -\frac{1}{11} \end{bmatrix}$$

b. $B = \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}$ $d = 2(8) - (-4)(-4)$
 $16 - 16 = 0$
 no inverse for matrix B

The use of inverses to solve systems of equations is advantageous when we are required to solve more than one system of equations $AX = B$, involving the same coefficient matrix, A , and different matrices of constants, B .

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$