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Problem Solving Strategies Part II

Here we present a more advanced problem solving strategy: algebra. Beyond basic arithmetic, algebra is the basic language of mathematics. That is, algebra is the standard way to present, describe, and work with mathematical concepts. Though algebra is typically not directly taught in the elementary classroom, many algebraic ideas are presented in many contexts. In fact, we've already seen a lot of (informal) algebra in the topics we've covered:

- Solving $a \div b$ with the missing factor model of division is really an algebra problem.
- Solving $a - b$ with the missing addend model is another example of an algebra problem.
- The statement "the n-th odd number is $2n - 1$" is an example of an algebraic statment.

Variables. A **variable** is a symbol which represents a number (usually a number we don't know yet).

Variables help us to organize mathematics problems. One example is the problem where we discoverd that "the n-th odd number is $2n - 1$". Another example:

Example. What is the sum of the first 100 natural numbers?

We need to add the first 100 natural numbers. We could do this by actually adding all the numbers up, however, this will take way too long. Another plan is to use a variable. How do we choose a variable? Well, what we are trying to find is the sum of the first 100 natural numbers. So a good choice may be to let S stand for the value of this sum. Now we are going to do a little work to see if we can find an easy way to solve this problem.

First, we notice that:

$$S = 1 + 2 + 3 + ... + 100$$

but by commutativity of addition this is the same as

$$S = 100 + 99 + 98 + ... + 1$$

where we just add up the numbers in reverse order.

Now I can always add two equations and if I add these two equations in the right way:

$$2s = \underbrace{101 + 101 + 101 + ... + 101}_{100 \text{ terms}}.$$

But this simplifies

$$2s = 100 \times 101$$

and so

$$s = \frac{100 \times 101}{2} = \frac{2 \times 50 \times 101}{2} = 50 \times 101 = 5050.$$

Thus the sum of the first 100 natural numbers is 5050.

Example. Example 1.10 in text.

Example. Your friend is worried about her grade in her biology course. Since she knows you are in MA 201, she asks you to use Polya's principles to help her figure out what score she needs on the final to get a C-. Her 4 exam grades were 48, 60,75, and 72 and exams are the only component of her grade.

The problem is to find the score your friend needs on the final exam for a 70 average given that the first four exams scores were 48,60,75, and 72.

The plan is to use a variable.

Since what we don't know is the final exam score we will let this be our variable and call it F. We now must setup an equation. Well, what we need is for the average of all the scores on the exams and the final to be a 75. To get the average of 5 numbers we

add them up and divide by 5, so the equation should be:

$$\frac{48+60+75+72+F}{5}=70.$$

Now since we need the final exam score, we must solve for F.

Multiply both sides by 5:

$$48+60+75+72+F=5\times 70$$

Simplify:

$$255+F=350.$$

Subtract 255 from both sides:

$$F=95.$$

So your friend must score a 95 on the exam.

Example. It takes you 2 hours to clean the attic. It takes your brother 5 hours. If you work together, how long will it take to clean the attic?

We need to find the total time it will take to clean the attic. Plan: use a variable. Since the total time is what we need to find, this should be our variable. So T is the total time.

Now since you can clean the attic in two hours, you can clean half of it per one hour. So in T hours, you get $\frac{1}{2}\cdot T$ of the attic cleaned. Your brother can clean the attic in 5 hours, so $\frac{1}{5}$ of it per one hour. In T hours your brother can clean $\frac{1}{5}\cdot T$ of the attic.

We need the whole attic clean so:

$$\frac{T}{2}+\frac{T}{5}=1$$

Multiply both sides of the equation by the common denominator 10:

$$\frac{10}{2}T+\frac{10}{5}T=10$$

and simplify

$$5T+2T=10.$$