Chapter 6: Indeterminate Structures – Direct Stiffness Method

1. Introduction

- *Force method* and *slope-deflection* method can be used, with hand calculation, for solving the indeterminate structures when the degree of static or kinematical indeterminacy is small.

- In this chapter, *direct stiffness method* (which is also called the *displacement method*) will be introduced that is a modern method for structural analysis. *Statically determinate and indeterminate problems can be solved in the same way*. The most important characteristic is the ability to automate the solution process so that implementation in a computer program is possible. Its methodology forms the backbone of the modern *finite element method*-based commercial programs that are used routinely to analyze a variety of structural systems.

2. Fundamentals of Matrix Algebra

- ***Definitions***:

  Matrix $A_{m\times n}$, Square matrix $A_{m\times m}$, Vector $a_{1\times n}$ (row vector) or $a_{m\times 1}$ (column vector).

- ***Operations***:

  Addition and subtraction:

  If $A_{m\times n} = B_{m\times n} \pm C_{m\times n}$ then $A_{ij} = B_{ij} \pm C_{ij} \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$.

  Multiplication:

  If $A_{m\times n} = B_{m\times k} C_{k\times n}$ then $A_{ij} = \sum_{l=1}^{k} B_{il} C_{lj}$.

  Transpose: $A^T_{n\times m}$.

Determinant: $\det(A_{m\times m})$.

Inverse: $A_{m\times m}^{-1}$.

Solution of linear algebraic equations:

$A_{m\times n}x_{n\times 1} = b_{m\times 1}$.

3. Basic Procedure of the Stiffness Method

Use a two-member spring example to illustrate these steps.

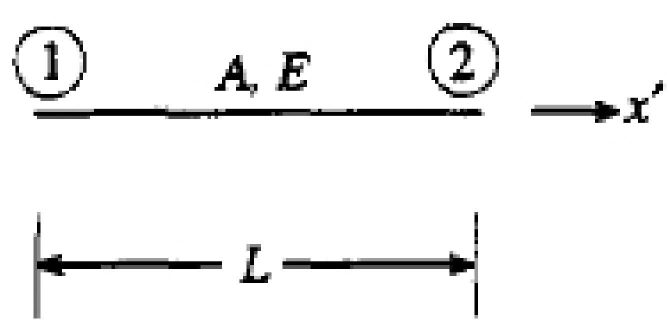
- Assign a coordinate system for the structure. Assign node numbers for the structure.
- Define degrees of freedom for the structure and assign numbers for them.
- Assign numbers to the members of the structure.
- Break-up the structure into smaller pieces called elements or members. Define element nodal displacements and forces.
- Identify boundary conditions for the structure in terms of displacements.
- Write compatibility conditions between the structural nodal displacements and element nodal displacements for each member.
- Identify external loads for each degree of freedom
- Write equilibrium equations for each element in terms of displacements. **kd = f**
- Combine elements equilibrium equations to form equilibrium equations for the entire structure. **KD = F**
- Apply boundary conditions to the system equilibrium equations and solve for the system nodal displacements.
- Finally, consider equilibrium equation for each element and solve for the element nodal forces.

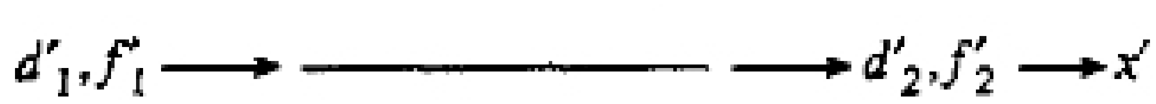
4. Direct Stiffness Method for Truss Analysis

- A truss is a structural system that satisfies the following requirements:
  a. The members are straight, slender, and prismatic. The cross-sectional dimensions are small in comparison to the member lengths. The weights of the members are small compared to the applied loads and can be neglected. Also when constructing the truss model for analysis, we treat the members as a one-dimensional entity (having length and negligible cross-sectional dimensions).
  b. The joints are assumed to be frictionless pins (or internal hinges).
  c. The loads are applied only at the joints in the form of concentrated forces.

As a consequence of these assumptions, the members are two-force members, meaning that they carry only axial forces. In very many ways, a truss member is quite similar to the typical linear spring. Two nodes define a typical truss element.

- For a two-node truss element shown below, governing equations with respect to the local coordinate system x' can be expressed as:





$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1' \\ d_2' \end{Bmatrix} = \begin{Bmatrix} f_1' \\ f_2' \end{Bmatrix} \quad \text{or} \quad \mathbf{k}'_{2\times2} \boldsymbol{d}'_{2\times1} = \boldsymbol{f}'_{2\times1}.$$

- From the transformation matrix between the local and global coordinate systems shown below, the relationship between the local nodal displacements and global nodal displacements is derived as