Review Material for the Final Examination of Math 21b
January 17, 2008

(1) Coefficient matrix and augmented matrix of a system of linear equations.

(2) Reduced row-echelon form of a matrix characterized by three conditions:

a. If a row has nonzero entries, then the first nonzero entry is 1, called the leading 1 in this row.

b. If a column contains a leading 1, then all other entries in that column are zero.

c. If a row contains a leading 1, then each row above contains a leading 1 further to the left.

(3) Reduction of a matrix to reduced row-echelon form by using three kinds of row operations (swapping rows, multiplying a row by a nonzero number, and adding a multiple of a row to another).

(4) Use reduction to reduced row-echelon form to determine whether a system of linear equations is inconsistent, uniquely solvable, or solvable with an infinite number of solutions and to give a general solution (if it exists) by using free variables.

(5) Use reduction to reduced row-echelon form to determine whether a square matrix is invertible and to find its inverse if it is invertible.

(6) Determinant of a 2×2 matrix. Formula for the inverse matrix of a 2×2 matrix with nonzero determinant.

(7) The span of a set of vectors. Redundant vectors in a sequence of vectors. Linear dependence and independence of a set of vectors. Subspaces. Bases. Dimension.

(8) Determine the rank and the nullity of a matrix. Find a basis for the image and for the kernel of a matrix. Rank-Nullity Theorem.

(9) Equivalence conditions for the invertibility of an $n \times n$ matrix A: unique solvability of $A\vec{x} = \vec{b}$, $\mathrm{rref}(A) = I_n$, $\mathrm{rank}(A) = n$, $\mathrm{im}(A) = \mathbb{R}^n$, $\ker(A) = \{\vec{0}\}$, column vectors forming a basis of $\mathbb{R}^n$, column vectors spanning $\mathbb{R}^n$, column vectors linearly independent.

(10) Special linear transformations: rotations, dilations, projections (onto a line or a plane), reflections, and shears (horizontal and vertical).

(11) The column vectors of the matrix of a linear transformation equal to its images of the standard vectors.

(12) Relation between matrix multiplication and the composition of linear transformations.

(13) Coordinates with respect a basis of a subspace. Matrix of a linear transformation with respect to a basis. Relation of matrices of the same linear transformation with respect to two different bases. Similar matrices. Powers of similar matrices. Similarity as an equivalence relation. The vector x and and its new coordinates $\vec{y}$ with respect to a basis $\vec{v}_1, \cdots, \vec{v}_n$ are related by

$$\vec{x} = \sum_{k=1}^{n} y_j \vec{v}_j = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = S\vec{y},$$

where

$$S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}.$$

A matrix A of a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ is related to the new matrix

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}.$$

representing T with respect to $\vec{v}_1, \cdots, \vec{v}_n$ by $AS = SB$, because

$$A\vec{v}_k = \sum_{k=1}^{n} b_{jk} \vec{v}_j = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} b_{1k} \\ \cdot \\ \cdot \\ \cdot \\ b_{nk} \end{bmatrix}$$

and

$$A\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix} = \sum_{k=1}^{n} b_{jk}\vec{v}_j = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}\begin{bmatrix}b_{11} & \cdots & b_{1n}\\ \cdot & \cdots & \cdot\\ \cdot & \cdots & \cdot\\ \cdot & \cdots & \cdot\\ b_{n1} & \cdots & b_{nn}\end{bmatrix}.$$

(14) Concept of a linear space (also known as a vector space). Addition and scalar multiplication in a linear space and the laws (associativity, commutativity, distributivity, etc.) satisfied by them. Examples of linear spaces: solutions of differential equations, spaces of polynomials, spaces of matrices, etc. Dimension of linear space. Finite and infinite dimension.

(15) Gram-Schmidt process of inductively constructing orthonormal vectors $u_1, \cdots, \vec{u}_m$ from linearly independent vectors $\vec{v}_1, \cdots, \vec{v}_m$ in $\mathbb{R}^n$. $\vec{v}_j^{\perp}$ is the orthogonal projection onto the subspace spanned by $\vec{v}_1, \cdots, \vec{v}_{j-1}$ (which is the same as the subspace spanned by $\vec{u}_1, \cdots, \vec{u}_{j-1}$) and $\vec{u}_j$ is the unit vector in the direction of $\vec{v}_j^{\perp}$.

(16) QR decomposition of an $n \times m$ matrix A in the form QR, where Q is an $n \times m$ matrix whose column vectors are orthonormal and R is an $m \times m$ matrix which is upper triangular with positive diagonal entries.

$$\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_m\end{bmatrix} = \begin{bmatrix}\vec{u}_1 & \cdots & \vec{u}_m\end{bmatrix} R,$$

where R is the upper triangular $m \times m$ matrix whose (i, j)-th entry is $r_{i,j} = \vec{u}_i \cdot \vec{v}_j$.

(17) Orthogonal transformations as length-preserving and orthogonality-preserving transformations. Orthonormal set of vectors and orthonormal basis. Pythagorean theorem. Cauchy-Schwarz inequality. Angle between vectors.

(18) Transpose of a matrix. Product of transposes of matrices and inverse of the transpose of a matrix. Symmetric and skew-symmetric matrices. Inner product of two vectors as the matrix product of the transpose of a vector and the other vector: $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$. The kernel of the transpose of a matrix as orthogonal complement of the image of the matrix: $(\mathrm{im}(A))^{\perp} = \ker(A^T)$.