7

Specifications

7.1 Introduction

In this chapter we will discuss how the properties of a control system can be specified. This is important for control design because it gives the goals. It is also important for users of control so that they know how to specify, evaluate and test a system so that they know it will have the desired properties. Specifications on a control systems typically include: stability of the closed loop system, robustness to model uncertainty, attenuation of measurement noise, injection of measurement noise, and ability to follow reference signals. From the results of Chapter 5 it follows that these properties are captured by six transfer functions called the Gang of Six. The specifications can be expressed in terms of these transfer functions. Essential features of the transfer functions can be expressed in terms of their poles and zeros or features of time and frequency responses.

7.2 Stability and Robustness to Process Variations

Stability and robustness to process uncertainties can be expressed in terms of the loop transfer function $L = PC$, the sensitivity function and the complementary sensitivity function

$$S = \frac{1}{1+PC} = \frac{1}{1+L}, \qquad T = \frac{PC}{1+PC} = \frac{L}{1+L}.$$

Since both S and T are functions of the loop transfer function specifications on the sensitivities can also be expressed in terms of specifications on the loop transfer function L. Many of the criteria are based on Nyquist's

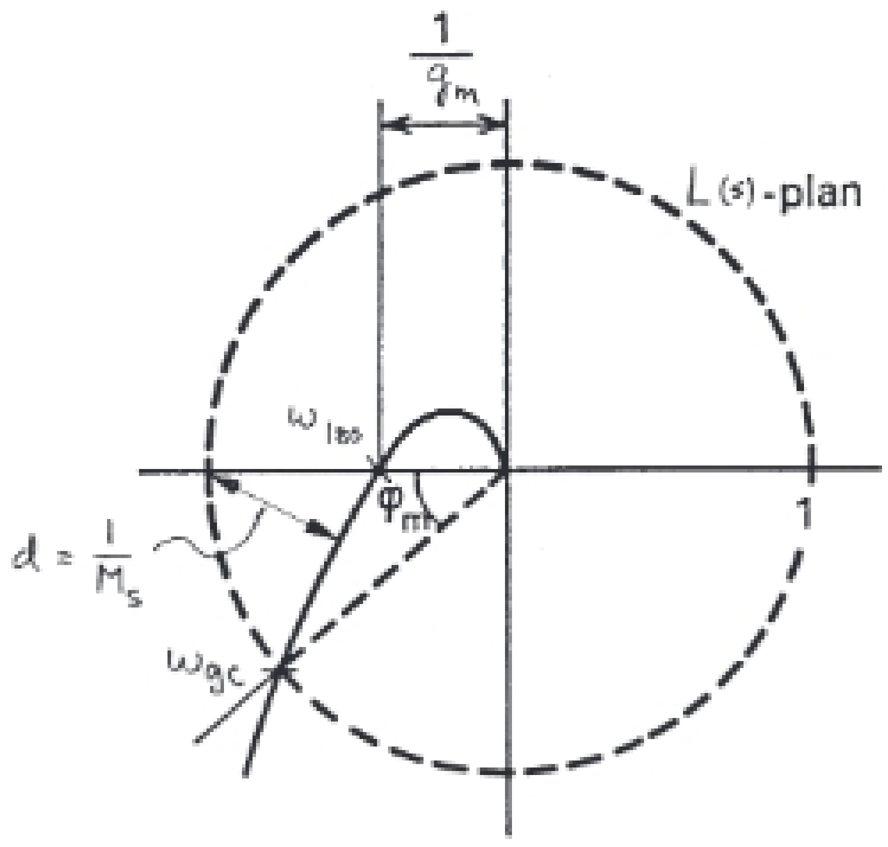

Figure 7.1 Nyquist curve of the loop transfer function L with indication of gain, phase and stability margins.

stability criterion, see Figure 7.1. Common criteria are the maximum values of the sensitivity functions, i.e.

$$M_s = \max_{\omega} |S(i\omega)|, \quad M_t = \max_{\omega} |T(i\omega)|$$

Recall that the number $1/M_s$ is the shortest distance of the Nyquist curve of the loop transfer function to the critical point, see Figure 7.1. Also recall that the closed loop system will remain stable for process perturbations ΔP provided that

$$\frac{|\Delta P(i\omega)|}{|P(i\omega)|} \leq \frac{1}{|T(i\omega)|},$$

see Section 5.5. The largest value M_t of the complementary sensitivity function T is therefore a simple measure of robustness to process variations.

Typical values of the maximum sensitivities are in the range of 1 to 2. Values close to one are more conservative and values close to 2 correspond to more aggressive controllers.

Gain and Phase Margins

The gain margin g_m and the phase margin ϕ_m are classical stability criteria. Although they can be replaced by the maximum sensitivities it is useful to know about them because they are still often used practically.

The gain margin tells how much the gain has to be increased before the closed loop system becomes unstable and the phase margin tells how much the phase lag has to be increased to make the closed loop system unstable.

The gain margin can be defined as follows. Let ω_{180} be the lowest frequency where the phase lag of the loop transfer function $L(s)$ is 180°. The gain margin is then

$$g_m = \frac{1}{|L(i\omega_{180})|} \tag{7.1}$$

The phase margin can be defined as follows. Let ω_{gc} denote gain crossover frequency, i.e. the lowest frequency where the loop transfer function $L(s)$ has unit magnitude. The phase margin is then given by

$$\phi_m = \pi + \arg L(i\omega_{gc}) \tag{7.2}$$

The margins have simple geometric interpretations in the Nyquist diagram of the loop transfer function as is shown in Figure 7.1. Notice that an increase of controller gain simply expands the Nyquist curve radially. An increase of the phase of the controller twists the Nyquist curve clockwise, see Figure 7.1.

Reasonable values of the margins are phase margin $\phi_m = 30° - 60°$, gain margin $g_m = 2 - 5$. Since it is necessary to specify both margins to have a guarantee of a reasonable robustness the margins g_m and ϕ_m can be replaced by a single stability margin, defined as the shortest distance of the Nyquist curve to the critical point -1, this distance is the inverse of the maximum sensitivity M_s. It follows from Figure 7.1 that both the gain margin and the phase margin must be specified in order to ensure that the Nyquist curve is far from the critical point. It is possible to have a system with a good gain margin and a poor phase margin and vice versa. It is also possible to have a system with good gain and phase margins which has a poor stability margin. The Nyquist curve of the loop transfer function of such a system is shown in Figure 7.2. This system has infinite gain margin, a phase margin of 70° which looks very reassuring, but the maximum sensitivity is $M_s = 3.7$ which is much too high. Since it is necessary to specify both the gain margin and the phase margin to endure robustness of a system it is advantageous to replace them by a single number. A simple analysis of the Nyquist curve shows that the following inequalities hold.

$$\begin{aligned} g_m &\geq \frac{M_s}{M_s - 1} \\ \varphi_m &\geq 2 \arcsin \frac{1}{2M_s} \end{aligned} \tag{7.3}$$