6.003: Signals and Systems

Fourier Representations

March 30, 2010

Mid-term Examination #2

Wednesday, April 7, 7:30-9:30pm, 34-101.

No recitations on the day of the exam.

Coverage: Lectures 1–15
Recitations 1–15
Homeworks 1–8

Homework 8 will not collected or graded. Solutions will be posted.

Closed book: 2 pages of notes ($8\frac{1}{2} \times 11$ inches; front and back).

Designed as 1-hour exam; two hours to complete.

Review sessions during open office hours.

Conflict? Contact freeman@mit.edu before Friday, April 2, 5pm.

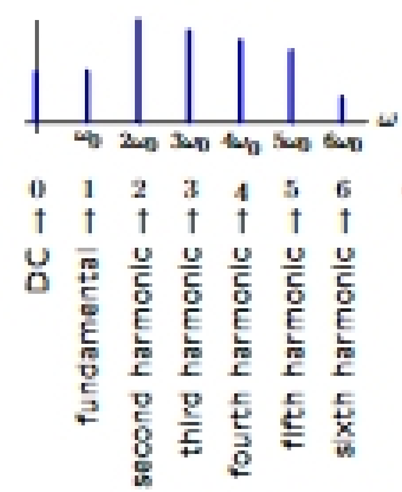
Fourier Representations

Fourier series represent **signals** in terms of **sinusoids**.

→ leads to a new representation for **systems** as **filters**.

Fourier Series

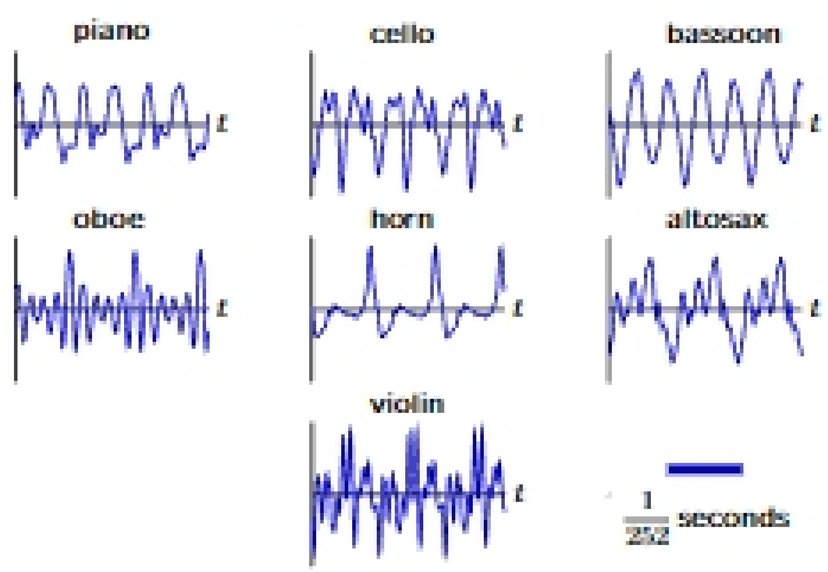
Representing signals by their harmonic components.



Musical Instruments

Harmonic content is natural way to describe some kinds of signals.

Ex: musical instruments (http://theremin.music.uiowa.edu/MIS)



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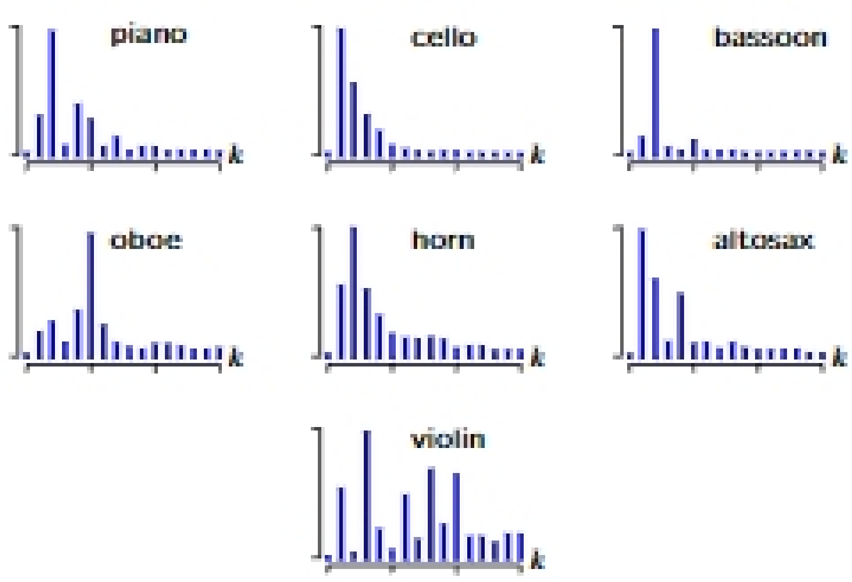

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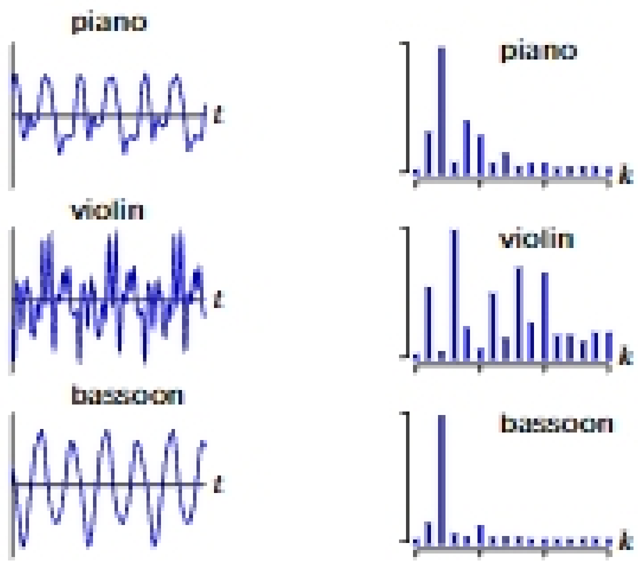
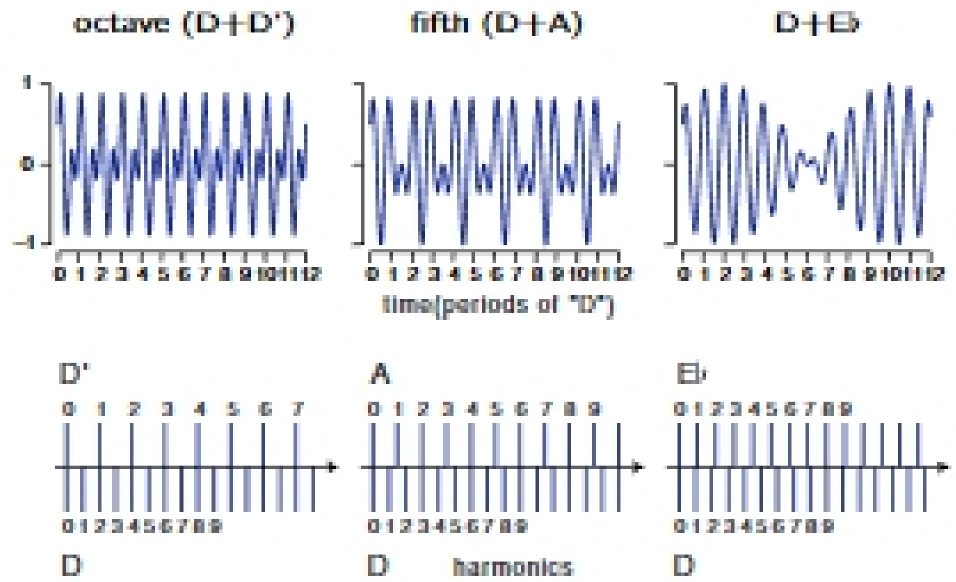


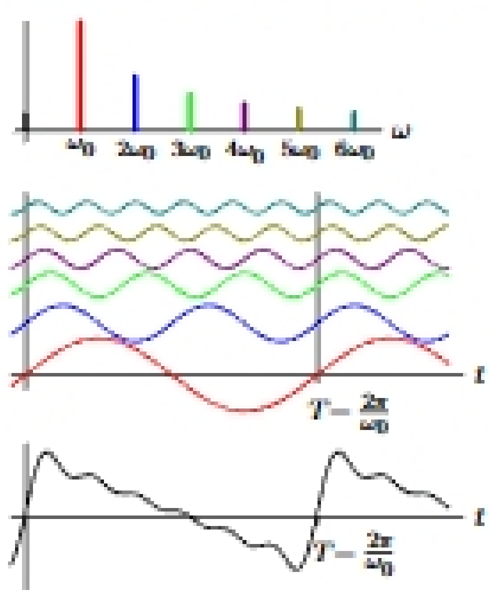
Harmonics

Harmonic structure determines consonance and dissonance.



Harmonic Representations

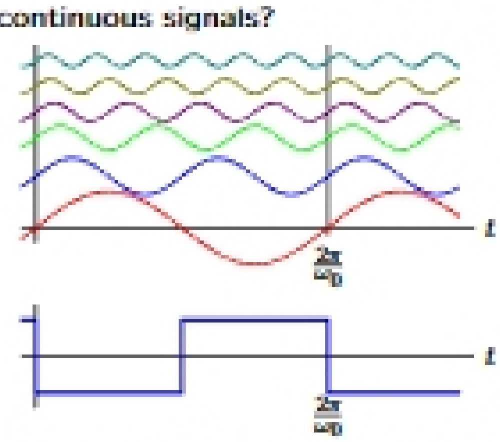
What signals can be represented by sums of harmonic components?



Only periodic signals: all harmonics of ω_0 are periodic in $T = 2\pi/\omega_0$.

Harmonic Representations

Is it possible to represent ALL periodic signals with harmonics? What about discontinuous signals?



Fourier claimed YES — even though all harmonics are continuous!

Lagrange ridiculed the idea that a discontinuous signal could be written as a sum of continuous signals.

We will assume the answer is YES and see if the answer makes sense.

Separating harmonic components

Underlying properties.

1. Multiplying two harmonics produces a new harmonic with the same fundamental frequency:

$$e^{jk\omega_0 t} \times e^{jl\omega_0 t} = e^{j(k+l)\omega_0 t}.$$

2. The integral of a harmonic over any time interval with length equal to a period T is zero unless the harmonic is at DC:

$$\int_{t_0}^{t_0+T} e^{jk\omega_0 t}dt \equiv \int_T e^{jk\omega_0 t}dt = \begin{cases} 0, & k \neq 0 \\ T, & k = 0 \end{cases}$$
$$= T\delta[k]$$

Separating harmonic components

Assume that $x(t)$ is periodic in T and is composed of a weighted sum of harmonics of $\omega_0 = 2\pi/T$.

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

Then

$$\int_T x(t)e^{-jl\omega_0 t}dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} e^{-j\omega_0 lt}dt$$
$$= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_0(k-l)t}dt$$
$$= \sum_{k=-\infty}^{\infty} a_k T\delta[k-l] = Ta_l$$

Therefore

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\omega_0 kt}dt \quad = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$

Fourier Series

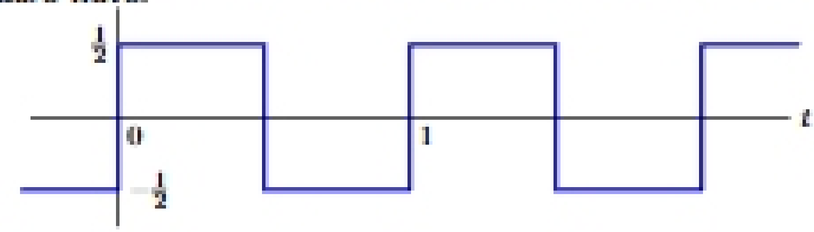
Determining harmonic components of a periodic signal.

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt \qquad \text{("analysis" equation)}$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \qquad \text{("synthesis" equation)}$$

Check Yourself

Let a_k represent the Fourier series coefficients of the following square wave.



How many of the following statements are true?

1. $a_k = 0$ if k is even
2. a_k is real-valued
3. $|a_k|$ decreases with k^2
4. there are an infinite number of non-zero a_k
5. all of the above

Fourier Series Properties

If a signal is differentiated in time, its Fourier coefficients are multiplied by $j\frac{2\pi}{T}k$.

Proof: Let

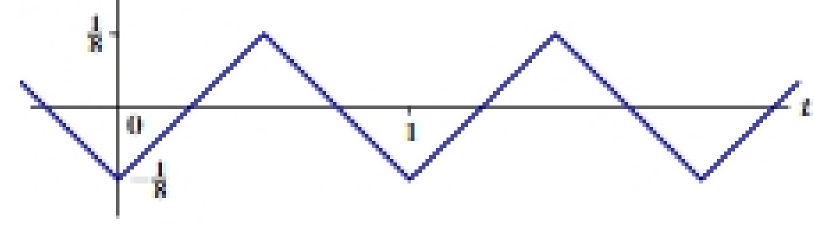
$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

then

$$\dot{x}(t) = \dot{x}(t+T) = \sum_{k=-\infty}^{\infty} \left(j\frac{2\pi}{T}k\,a_k\right) e^{j\frac{2\pi}{T}kt}$$

Check Yourself

Let b_k represent the Fourier series coefficients of the following triangle wave.



How many of the following statements are true?

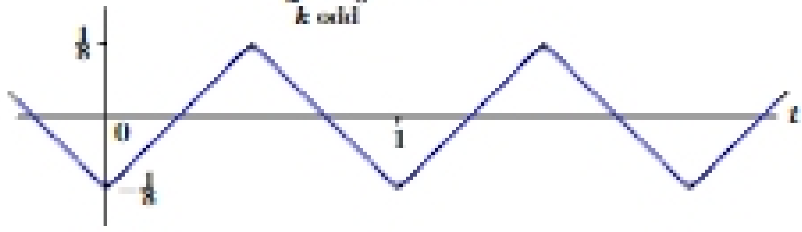
1. $b_k = 0$ if k is even
2. b_k is real-valued
3. $|b_k|$ decreases with k^2
4. there are an infinite number of non-zero b_k
5. all of the above

Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

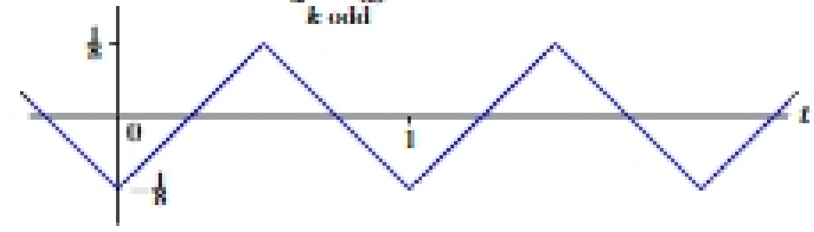


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



Fourier series representations of functions with discontinuous slopes converge toward functions with discontinuous slopes.