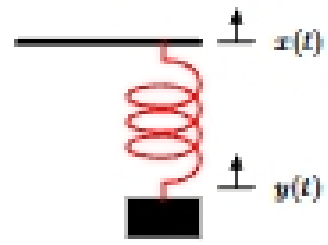
6.003: Signals and Systems

Second-Order Systems

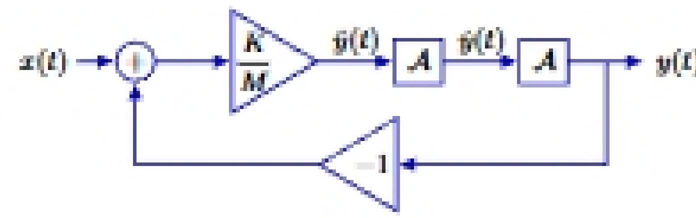
October 8, 2009

Last Time

We analyzed a mass and spring system.



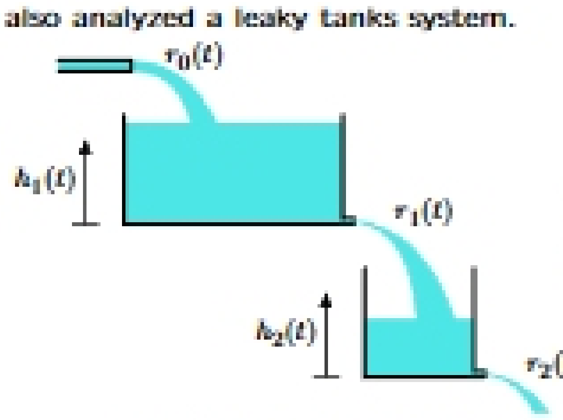
$$F = K\left(x(t) - y(t)\right) = M\ddot{y}(t)$$



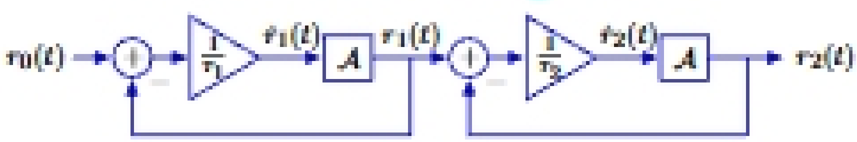
$$\frac{Y}{X} = \frac{\frac{K}{M}\mathcal{A}^2}{1+\frac{K}{M}\mathcal{A}^2}$$

Last Time

We also analyzed a leaky tanks system.



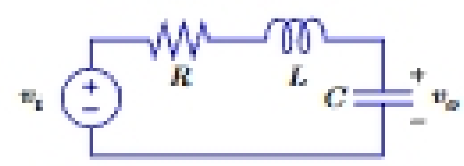
$$\tau_1\dot{r}_1(t) = r_0(t) - r_1(t)$$
$$\tau_2\dot{r}_2(t) = r_1(t) - r_2(t)$$



$$\frac{R_2}{R_0} = \frac{\mathcal{A}/\tau_1}{1+\mathcal{A}/\tau_1} \times \frac{\mathcal{A}/\tau_2}{1+\mathcal{A}/\tau_2}$$

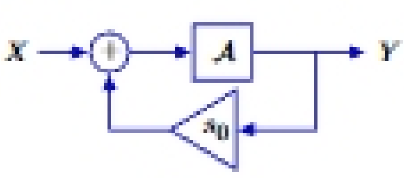
Second-Order Systems

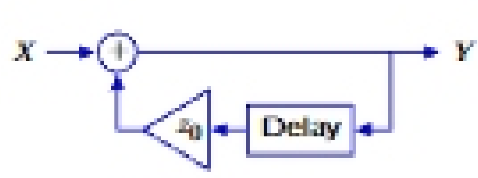
Today: Look more carefully at growth and decay of oscillatory responses by studying an analogous electrical circuit.



But First ...

The canonical forms for CT and DT differ.

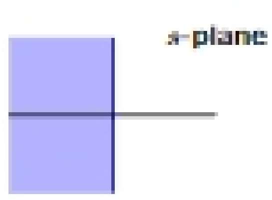


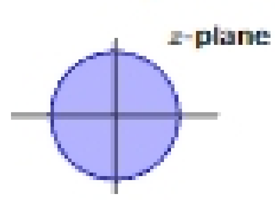


$$H = \frac{Y}{X} = \frac{\mathcal{A}}{1-s_0\mathcal{A}} \qquad H = \frac{Y}{X} = \frac{1}{1-z_0\mathcal{R}}$$

$$h(t) = e^{s_0t}u(t) \qquad h[n] = z_0^n u[n]$$

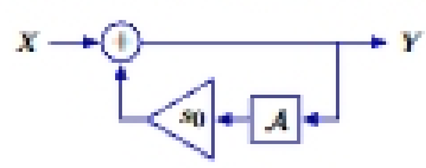
$$\mathcal{A} \rightarrow \frac{1}{s} \qquad \mathcal{R} \rightarrow \frac{1}{z}$$





Check Yourself

What if we had used the DT canonical form for CT?



What is the impulse response of this system?

1. $s_0e^{s_0t}u(t)$
2. $s_0^t u(t)$
3. $1 + s_0e^{s_0t}u(t)$
4. $\delta(t) + s_0e^{s_0t}u(t)$
5. none of the above

Second-Order Systems

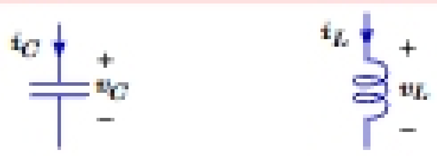
Today: Look more carefully at growth and decay of oscillatory responses by studying an analogous electrical circuit.

Second-Order Systems

Solve with state variable approach.

State variables represent the minimum knowledge of the past ($t < t_0$) needed to propagate the output into the future ($t > t_0$).

Check Yourself



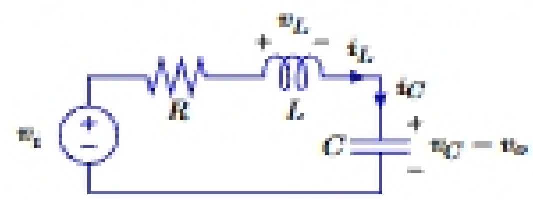
$i_C = C\frac{dv_C}{dt}$ $\qquad$ $v_L = L\frac{di_L}{dt}$

Which of the following can be state variables?

1. v_C and v_L
2. i_C and v_L
3. i_C and i_L
4. v_C and i_L
5. i_C and v_C and i_L and v_L
6. none of above

Second-Order Systems

State variable approach: determine expressions for derivatives of state variables in terms of (undifferentiated) state variables.

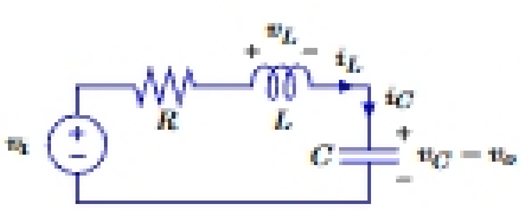


$$\frac{dv_C}{dt} = \frac{1}{C}i_C = \frac{1}{C}i_L \quad \text{(KCL)}$$

$$\frac{di_L}{dt} = \frac{1}{L}v_L = \frac{1}{L}(v_i - Ri_L - v_C) \quad \text{(KVL)}$$

Second-Order Systems

Determine the system functional.



$$\frac{dv_C}{dt} = \frac{1}{C}i_L$$

$$\frac{di_L}{dt} = \frac{1}{L}(v_i - Ri_L - v_C)$$

Use first equation to eliminate i_L from the second equation:

$$C\frac{d^2v_C}{dt^2} = \frac{1}{L}\left(v_i - RC\frac{dv_C}{dt} - v_C\right).$$

Integrate twice (ignoring initial conditions ... why?)

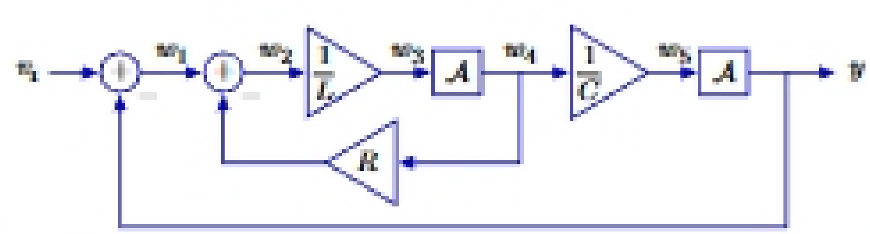
$$Cv_C = \frac{1}{L}(\mathcal{A}^2 v_i - RC\mathcal{A}v_C - \mathcal{A}^2 v_C)$$

$$\frac{V_o}{V_i} = \frac{V_C}{V_i} = \frac{\frac{\mathcal{A}^2}{LC}}{1 + \frac{R}{L}\mathcal{A} + \frac{1}{LC}\mathcal{A}^2}$$

Check Yourself

Alternatively, determine system functional from block diagram.

$$\frac{dv_C}{dt} = \frac{1}{C}i_L \qquad \frac{di_L}{dt} = \frac{1}{L}(v_i - Ri_L - v_C)$$



Which node corresponds to i_L?

1. w_1 2. w_2 3. w_3 4. w_4 5. w_5 6. y
7. none of the above

Second-Order Systems

Analogous systems.

RLC circuit

$$\frac{V_o}{V_i} = \frac{\frac{1}{LC}\mathcal{A}^2}{1+\frac{R}{L}\mathcal{A}+\frac{1}{LC}\mathcal{A}^2} = \frac{\omega_0^2\mathcal{A}^2}{1+\frac{\omega_0}{Q}\mathcal{A}+\omega_0^2\mathcal{A}^2} \qquad \omega_0=\sqrt{\frac{1}{LC}}$$

Mass and spring

$$\frac{Y}{X} = \frac{\frac{K}{M}\mathcal{A}^2}{1+\frac{K}{M}\mathcal{A}^2} = \frac{\omega_0^2\mathcal{A}^2}{1+\frac{\omega_0}{Q}\mathcal{A}+\omega_0^2\mathcal{A}^2} \qquad \omega_0=\sqrt{\frac{K}{M}}$$

Leaky tanks

$$\frac{R_2}{R_0} = \frac{\frac{1}{\tau_1\tau_2}\mathcal{A}^2}{1+\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)\mathcal{A}+\frac{1}{\tau_1\tau_2}\mathcal{A}^2} = \frac{\omega_0^2\mathcal{A}^2}{1+\frac{\omega_0}{Q}\mathcal{A}+\omega_0^2\mathcal{A}^2} \qquad \omega_0=\sqrt{\frac{1}{\tau_1\tau_2}}$$

Second-Order Systems

The effect of ω_0 is to scale time.

$$H=\frac{\omega_0^2\mathcal{A}^2}{1+\frac{\omega_0}{Q}\mathcal{A}+\omega_0^2\mathcal{A}^2}$$

There is one ω_0 paired with each $\mathcal{A}$ $\to$ let $\tilde{\mathcal{A}}=\omega_0\mathcal{A}$.

Then the system is reduced to a function of one parameter Q.

$$H=\frac{\tilde{\mathcal{A}}^2}{1+\frac{1}{Q}\tilde{\mathcal{A}}+\tilde{\mathcal{A}}^2}$$

Since $\mathcal{A}=\int_{-\infty}^{t}d\tau$, $\tilde{\mathcal{A}}=\int_{-\infty}^{\omega_0 t}d(\omega_0\tau)$.

Scaling

Scaling details.

$$\mathcal{A}x(t)=\int_{-\infty}^{t}x(\tau)d\tau \qquad \tilde{\mathcal{A}}y(\tau)=\int_{-\infty}^{\tau}y(\rho)d\rho$$

Want $x(t)=y(\omega_0 t)$.

Let $\rho=\omega_0\tau$
$\tau=\omega_0 t$
$d\rho=\omega_0 d\tau$

Then

$$\tilde{\mathcal{A}}y(\omega_0 t)=\int_{-\infty}^{\omega_0 t}y(\omega_0\tau)d(\omega_0\tau)=\omega_0\int_{-\infty}^{t}y(\omega_0\tau)d\tau=\omega_0\mathcal{A}y(\omega_0 t)$$

$$\tilde{\mathcal{A}}x(t)=\omega_0\mathcal{A}x(t)$$

$$\tilde{\mathcal{A}}=\omega_0\mathcal{A}$$

Second-Order Systems

Find the poles by factoring the denominator of the system functional.

$$H=\frac{\tilde{\mathcal{A}}^2}{1+\frac{1}{Q}\tilde{\mathcal{A}}+\tilde{\mathcal{A}}^2}=\frac{\tilde{\mathcal{A}}^2}{(1-\tilde{p}_0\tilde{\mathcal{A}})(1-\tilde{p}_1\tilde{\mathcal{A}})}.$$

It follows that

$$\tilde{p}_0+\tilde{p}_1=-\frac{1}{Q} \quad \text{and} \quad \tilde{p}_0\tilde{p}_1=1.$$

Alternatively, substitute $\tilde{\mathcal{A}}\to\frac{1}{\tilde{s}}$ and find roots of denominator:

$$\tilde{s}^2+\frac{1}{Q}\tilde{s}+1=0$$

Either way, the result is

$$\tilde{s}=\tilde{p}_0,\tilde{p}_1=-\frac{1}{2Q}\pm\sqrt{\left(\frac{1}{2Q}\right)^2-1}.$$

Second-Order Systems

Map pole locations as a function of Q.

If $Q\approx 0$, then $\frac{1}{Q}\gg 1$.

$$\tilde{s}^2+\underbrace{\frac{1}{Q}}_{\gg 1}\tilde{s}+1=0$$

Now if $|\tilde{s}|\ll 1$, the $\tilde{s}^2$ term can be neglected and $\tilde{s}\approx -Q$.

Or if $|\tilde{s}|\gg 1$, the $+1$ term can be neglected and $\tilde{s}\approx -\frac{1}{Q}$.

Second-Order Systems

For small Q, the poles are at $-Q$ and $-\frac{1}{Q}$.