COT3100 Spring'2001

Practice problems for the test#2

Relations.

1. Let A ={1, 2, 3}, B ={w, x, y, z}, and C = {4, 5, 6}. Define the relation $R \subseteq A \times B$, $S \subseteq B \times C$, and $T \subseteq B \times C$, where R = {(1, w), (3, w), (2, x), (1, y)}, S={(w, 5), (x, 6), (y, 4), (y, 6)}, and T={(w, 4), (w, 5), (y, 5)}.
   a) Determine $R \circ (S \cup T)$ and $(R \circ S) \cup (R \circ T)$.
   $S \cup T$ = {(w, 4), (w, 5), (x, 6), (y, 4), (y, 5), (y, 6)}, $R \circ (S \cup T)$ = {(1, 4), (1, 5), (1, 6), (2, 6), (3, 4), (3, 5)}
   $R \circ S$ = {(1, 4), (1, 5), (1, 6), (2, 6), (3, 5)}, $R \circ T$ = {(1, 4), (1, 5), (3, 4), (3, 5)}
   $(R \circ S) \cup (R \circ T)$ = {(1, 4), (1, 5), (1, 6), (2, 6), (3, 4), (3, 5)}.
   b) Determine $R \circ (S \cap T)$ and $(R \circ S) \cap (R \circ T)$.
   $S \cap T$ ={(w, 5)};
   $R \circ (S \cap T)$ ={(1, 5), (3, 5)};
   $(R \circ S) \cap (R \circ T)$ = {(1, 4), (1, 5), (3, 5)}.

2. Consider the relation T over real numbers: $T = \{(a, b) \mid a^2+b^2=1\}$. Define all properties of this relation (reflexive, irreflexive, symmetric, anti-symmetric, and transitive).

Not reflexive (for example, a =0.5, $(a, a) \notin T$).
Not irreflexive ($a=1/\sqrt{2}$, $(a, a) \in T$)
Symmetric (for any real a, b, $(a, b) \in T \Rightarrow (b, a) \in T$, because $a^2+b^2=1 \Rightarrow b^2+a^2=1$).
Not anti-symmetric (counterexample: $(0, 1) \in T$ and $(1, 0) \in T$)
Not transitive (counterexample: $(0, 1) \in T$, $(1, 0) \in T$, but $(0, 0) \notin T$)

3. Let $R \subseteq A \times A$ be a symmetric relation. Prove that $t(R)$ is symmetric.

Proof. Assume $R \subseteq A \times A$ is symmetric. To prove that $t(R)$ is symmetric, we need to show that if $(a, b) \in t(R)$ then $(b, a) \in t(R)$. So, pick arbitrary $(a, b) \in t(R)$ (1). By the definition of transitive closure, (1) implies that there is a path from a to b in R. By the definition of a path it means, that there are some elements $a_0=a$, a_1, a_2, … , $a_n=b \in A$ such that (a_0, a_1), (a_1, a_2), … $(a_{n-1}, a_n) \in R$. But since R is symmetric by assumption, all inverted pairs belong to R as well, i. e. (a_n, a_{n-1}), …, (a_2, a_1), $(a_1, a_0) \in R$. It signifies that R contains the path from $a_n=b$ to $a_0=a$. By the definition of the transitive closure it implies that $(b, a) \in t(R)$.

4. Prove or disprove the following propositions about arbitrary relations on A:
   a) R_1, R_2 symmetric $\Rightarrow R_1 \cup R_2$ is symmetric.
   Proof. Assume R_1 and R_2 are symmetric. To prove that $R_1 \cup R_2$ is symmetric we need to show that if $(a, b) \in R_1 \cup R_2$, then $(b, a) \in R_1 \cup R_2$. So, take arbitrary $(a, b) \in R_1 \cup R_2$. It implies that either $(a, b) \in R_1$ or $(a, b) \in R_2$. In the first case $(a, b) \in R_1$ implies $(b, a) \in R_1$ by symmetric property of R_1 and $(b, a) \in R_1 \cup R_2$ by the union property. In the second case, $(a, b) \in R_2$ implies $(b, a) \in R_2$ by symmetric property of R_2, and then $(b, a) \in R_1 \cup R_2$ by the union property. So, in both cases we showed that $(b, a) \in R_1 \cup R_2$.is implied.
   b) R_1, R_2 transitive $\Rightarrow R_1 \cup R_2$ is transitive.

Proof. Assume R_1 and R_2 are transitive. To prove that $R_1 \cap R_2$ is transitive, we need to show that if $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$ then $(a, c) \in R_1 \cap R_2$. So, assume that $(a, b) \in R_1 \cap R_2$ (1) and $(b, c) \in R_1 \cap R_2$ (2).
By the intersection definition we can imply from (1) that $(a, b) \in R_1$ and $(a, b) \in R_2$. Similar we can imply from (2) that $(b, c) \in R_1$ and $(b, c) \in R_2$. Then by the transitive property of R_1 we have that $(a, c) \in R_1$(3) and similar from the transitive property of R_2 we have that $(a, c) \in R_2$(4). (3) and (4) imply that $(a, c) \in R_1 \cap R_2$ by the definition of intersection. QED.

5. Prove or disprove that for any relation $R \subseteq A \times A$ $ts(R) \subseteq st(R)$.
The following counterexample can be used to disprove that $ts(R) \subseteq st(R)$.
Take $A = \{1, 2, 3\}$, $R = \{(1, 2), (2, 3)\}$, then $st(R) = \{(1, 2), (2, 1), (2, 3), (3, 2), (1, 3), (3, 1)\}$, $ts(R) = \{(1, 2), (2, 1), (2, 3), (3, 2), (1, 3), (3, 1), (1, 1), (2, 2), (3, 3)\}$ is not a subset of $st(R)$, since $(1, 1) \in ts(R)$ and $(1, 1) \notin st(R)$.

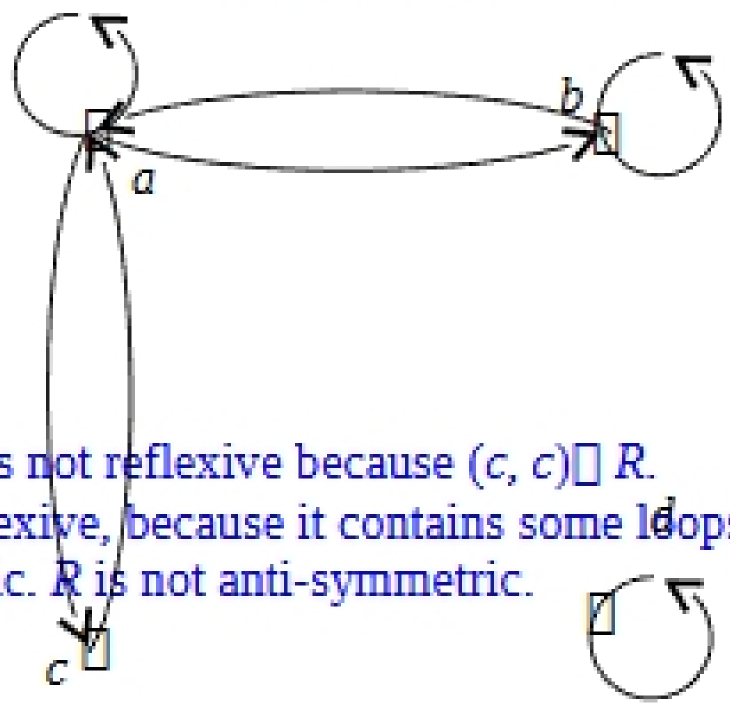
6. Let R be a symmetric and transitive relation on set S. Furthermore, suppose, that for every $x \in S$ there is an element $y \in S$ such that xRy. Prove that R is equivalence relation.
Proof. Assume R is a symmetric and transitive relation on set S and for every $x \in S$ there is an element $y \in S$ such that xRy. To prove that R is equivalence relation we need to prove that R is also reflexive, i.e. for every $x \in S$ xRx. Take any $x \in S$, then by assumption there exists an element $y \in S$ such that xRy. By symmetric property of R xRy implies yRx. By transitive property of R xRy and yRx imply xRx. So, we proved that for any $x \in S$ xRx.

5. Prove that a relation R on a set A is transitive if and only if $R^2 \subseteq R$.
We need to prove: i) if R is transitive, then $R^2 \subseteq R$. and ii) if $R^2 \subseteq R$. then R is transitive.

i) assume that R is transitive. To prove that $R^2 \subseteq R$. we need to show that if $(a, b) \in R^2$, then $(a, b) \in R$. So, take any $(a, b) \in R^2$. By the definition of R^2 as a composition R with itself it implies that there exists some $x \in A$ such that $(a, x) \in R$. and $(x, b) \in R$. By the transitive property of R, $(a, x) \in R$ and $(x, b) \in R$ imply $(a, b) \in R$. QED

ii) assume that $R^2 \subseteq R$.. To prove that R is transitive we need to show that if $(a, x) \in R$. and $(x, b) \in R$ then $(a, b) \in R$. So, assume that $(a, x) \in R$ and $(x, b) \in R$. Then by the definition of composition $(a, b) \in R \circ R = R^2$. Since $R^2 \subseteq R$ by assumption, $(a,b) \in R^2$ results in $(a, b) \in R$. QED.

6. Determine whether the relation represented by the directed graph is reflexive, irreflexive, symmetric, anti-symmetric and transitive.



The relation is not reflexive because $(c, c) \notin R$.
R is not irreflexive, because it contains some loops.
R is symmetric. R is not anti-symmetric.

R is not transitive, because $(c, b) \in R$ and $(b, c) \in R$, but $(c, c) \notin R$.

7. Let A denote an arbitrary non-empty set, and let R denote a binary relation, $R \subseteq A \times A$.
   Answer the following two parts **independently of each other**:
   (a) Suppose R is transitive. Prove that the inverse relation R^{-1} is also transitive, where R^{-1} is defined as $R^{-1} = \{(a, b) \mid (b, a) \in R\}$.

We need to show that if $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$, then $(a, c) \in R^{-1}$. Assume there are some $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$ (if not, R^{-1} is transitive and we have nothing to prove). By the definition of inverse relation $(a, b) \in R^{-1}$ implies $(b, a) \in R$ and $(b, c) \in R^{-1}$ implies $(c, b) \in R$. By the transitive property of R $(c, b) \in R$ and $(b, a) \in R$ imply $(c, a) \in R$. By the definition of inverse relation $(c, a) \in R$ implies $(a, c) \in R^{-1}$. QED.

   (b) Suppose $R \neq \emptyset$ and R is irreflexive (that is, there does **not** exist any $a \in A$ such that $(a, a) \in R$). Prove that either R is not symmetric or R is not transitive.

Proof by contradiction. Assume $R \neq \emptyset$ and R is irreflexive and R is symmetric and transitive. We are going to show that this leads to contradiction. $R \neq \emptyset$ implies that there exists some element $(x, y) \in R$. By symmetric property of R, $(x, y) \in R$ implies $(y, x) \in R$. By the transitive property of R $(x, y) \in R$ and $(y, x) \in R$ imply $(x, x) \in R$, that contradicts to the assumption that R is irreflexive. The contradiction proves that if $R \neq \emptyset$ and R is irreflexive, then either R is not symmetric or R is not transitive.

8. Let A denote an arbitrary non-empty set, and let R, S, and T denote binary relations defined over A, i.e., $R \subseteq A \times A$, $S \subseteq A \times A$, and $T \subseteq A \times A$. Answer the following two questions **independently of each of other**:
   (a) Prove $(R \circ (S \cap T)) \subseteq ((R \circ S) \cap (R \circ T))$.

We need to prove that any element from $(R \circ (S \cap T))$ belongs to $((R \circ S) \cap (R \circ T))$. So take arbitrary $(a, c) \in (R \circ (S \cap T))$ to show that $(a, c) \in ((R \circ S) \cap (R \circ T))$. By the definition of composite relation $(a, c) \in (R \circ (S \cap T))$ implies that there exists some $b \in A$ such that aRb and $b(S \cap T)c$ (i.e. $(b, c) \in (S \cap T)$). By the definition of intersection, $(b, c) \in (S \cap T)$ implies $(b, c) \in S$ and $(b, c) \in T$. By the idempotent law and the associative property of logical 'and', $(a, b) \in R$ and $((b, c) \in S$ and $(b, c) \in T)$ is equivalent to $((a, b) \in R$ and $(b, c) \in S)$ and $((a, b) \in R$ and $(b, c) \in T)$. By the definition of the composite relation this is equivalent to $(a, c) \in R \circ S$ and $(a, c) \in R \circ T$. By the definition of intersection $(a, c) \in R \circ S$ and $(a, c) \in R \circ T \Leftrightarrow (a, c) \in R \circ S \cap R \circ T$. QED

   (b) Suppose $A = \{a, b, c\}$. Use an example of relations R, S, and T defined over this A to show that $(R \circ (S \cap T)) \neq ((R \circ S) \cap (R \circ T))$.

The following counterexample disproves that $((R \circ S) \cap (R \circ T)) \subseteq (R \circ (S \cap T))$.
$R = \{(a, b), (a, c)\}$, $T = \{(b, a)\}$, $S \{(c, a)\}$. $((R \circ S) \cap (R \circ T)) = \{(a, a)\}$, $R \circ (S \cap T) = \emptyset$.

11. Let R and S be relations on X. Determine whether each statement is true or false. If the statement is false, give a counterexample.
   a) If R and S are transitive, then $R \cup S$ is transitive. False: $R = \{(a, b)\}$, $S = \{(b, c)\}$ are both transitive, but $R \cup S = \{(a, b), (b, c)\}$ is not transitive.
   b) If R and S are transitive, then $R \cap S$ is transitive. True.
   c) If R and S are transitive, then $R \circ S$ is transitive. True.
   d) If R is transitive, then R^{-1} is transitive. True.