10-21-2009

Divisibility Tests

There are many simple rules that can allow us to quickly check if one number is a divisor of another. In this short section we collect a few of these tests.

1. **Sum rule.** Suppose a and b are divisible by a third number c. Then $a+b$ is also divisible by c.

   Proof. If c divides a then $a = cq$ for some whole number q. If c also divides b then $b = cm$ for some whole number n. Now add a and b

   $$a + b = cq + cm = c(q + m).$$

   We know that $a+b$ is divisible by c if we can write $a+b$ as c times some other whole number. But we know that

   $$a + b = c(q + m)$$

   and since $q + m$ is a whole number $a + b$ is divisible by c. $\square$

2. **Difference rule.** Suppose a and b are divisible by a third number c. Then $a - b$ is also divisible by c.

3. **Divisibility by 2.** A number is divisible by 2 if its last digit is 0,2,4,6, or 8.

4. **Divisibility by 5.** A number is divisible by 5 if its last digit is 0 or 5.

5. **Divisibility by 10.** A number is divisible by 10 if its last digit is 0.

6. **Divisibility by 4.** A number is divisible by 4 when its last two digits form a number divisible by 4.

   Proof. Given any number n, we can perform division with remainder to get

   $$n = q \cdot 100 + r$$

where r is the number formed by the last two digits of n.

Now certainly, $q \cdot 100$ is divisible by 4 since $(q \cdot 100) \div 4 = q \cdot 25 \quad R0$.

If r is also a number divisible by 4 we can use the sum rule to say that $n = q \cdot 100 + r$ is divisible by 4 too.

So whenever r (the number formed by the last two digits of n) is divisible by 4, so is the number n. □

7. **Divisibility by 8.** A number is divisible by 8 when its last three digits form a number divisible by 8.

8. **Divisibility by 3.** A number is divisible by 3 if and only if the sum of its digits is a number divisible by 3.

9. **Divisibility by 9.** A number is divisible by 9 if and only if the sum of its digits is a number divisible by 9.

10. **Divisibility by 11.** A number is divisible by 11 when the sum of its digits in the even positions minus the sum of the digits in the odd positions is divisible by 11.

11. **Divisibility by 7.** Start at the right and group the digits of the number into blocks of 3. Number the blocks starting with 1 for the right most block, 2 for the next, and so on until you reach the left most digit block. Add up the even blocks. Add the odd blocks. Subtract the sum of the odd blocks from the sum of the even blocks. If this difference is divisible by 7, then so is the number.

12. **Divisibility by 13.** Start at the right and group the digits of the number into blocks of 3. Number the blocks starting with 1 for the right most block, 2 for the next, and so on until you reach the left most digit block. Add up the even blocks. Add up the odd blocks. Subtract the sum of the odd blocks from the sum of the even blocks. If this difference is divisible by 13, then so is the number.

Examples.

1. **Divisibility by 2.** The number 130,354,210,008 is divisible by 2 since the number ends in 8 and 8 is divisible by 2.

2. **Divisibility by 5.** The number 35,323,315 is divisible by 5 since its last digit is a 5.

3. **Divisibility by 10.** The number 35,323,310 is divisible by 10, 2, and 5 since its last digit is a 0.

4. **Divisibility by 4.** The number 35,323,336 is divisible by 4, since its last two digits give the number 36 which is divisible by 4.

5. **Divisibility by 8.** The number 35,323,088 is divisible by 8 since its last three digits give the number 088 which is divisible by 8.

6. **Divisibility by 3.** The number 35,323,143 is divisible by 3 since its digits sum to $3+5+3+2+3+1+4+3 = 8+8+8 = 3(8)$ which is divisible by 3.

7. **Divisibility by 9.** The number 35,333,811 is divisible by 9 since its digits sum to $3+5+3+3+3+8+1+1 = 9+9+9 = 9(3)$ which is divisible by 9.

8. **Divisibility by 11.** The number 35,838 is divisible by 11 since: the sum of the even position digits is $3+5 = 8$, the sum of its odd position digits is $8+8+3 = 19$, and $8 - 19 = -11$ which is divisible by 11.

9. **Divisibility by 7.** Consider the number 122,268,636. The sum of the odd blocks is 122+636=758. The sum of the even blocks is 268. The difference of the even block and odd block sum is -490 which is divisible by 7.

10. **Divisibility by 13.** Consider the number 141,057,004,608. The sum of the odd blocks 608+057=665. The sum of the even blocks is $004+141 = 145$. The difference in the even and odd blocks is 540 which is divisible by 13.