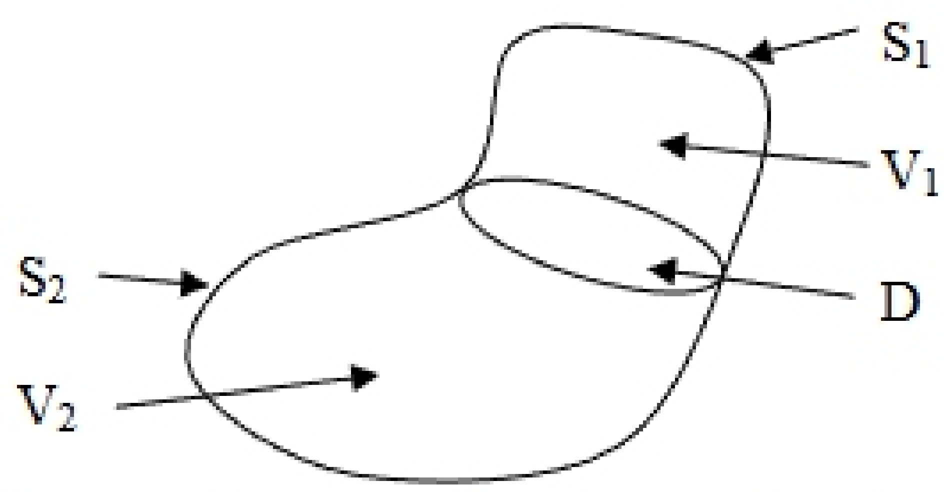
Vector Calculus Theorems

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Gauss' Theorem (Divergence Theorem)

Consider a surface S with volume V. If we divide it in half into two volumes V_1 and V_2 with surface areas S_1 and S_2, we can write:

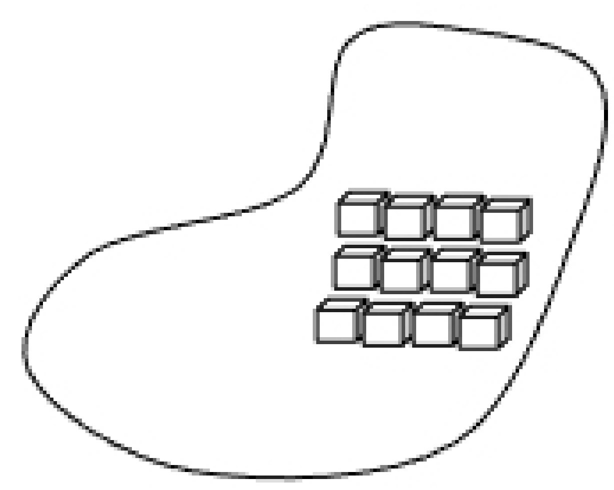
$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{A} = \oint_{S_1} \mathbf{E} \cdot d\mathbf{A} + \oint_{S_2} \mathbf{E} \cdot d\mathbf{A}$$



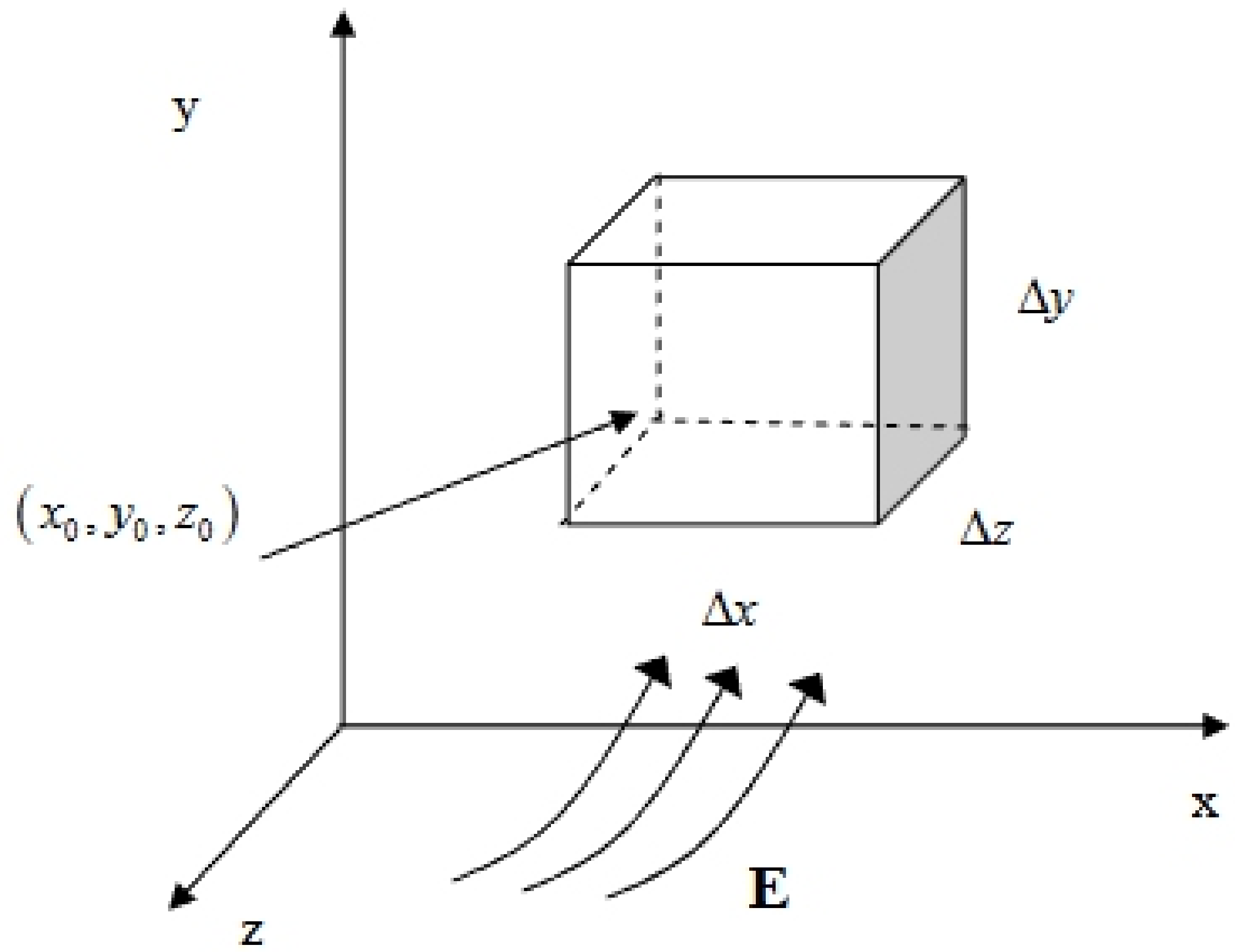
since the electric flux through the boundary D between the two volumes is equal and opposite (flux out of V_1 goes into V_2).

Now let's continue this process of dividing the original volume into a great number of infinitesimal volumes, each cubic in shape:

$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{A} = \sum_i \oint_{S_i} \mathbf{E} \cdot d\mathbf{A}$$



Consider now one of these small cubic volumes. Consider one corner of this cube at position (x_0, y_0, z_0). The length of each side is $\Delta x, \Delta y, \Delta z$, and each face is perpendicular to one of the coordinate axes.



We are interested in computing the flux passing through this small volume. The flux through the top and bottom faces will only depend on E_y since $\mathbf{E}\cdot d\mathbf{A}=0$ for the other 4 faces. Since the cube is infinitesimal, we can do a Taylor expansion of the field about (x_0, y_0, z_0) and find the y component of the field at the center of the bottom face $\left(x+\frac{\Delta x}{2}, y, z+\frac{\Delta z}{2}\right)$:

$$E_y^{\text{bottom}} = E_y\left(x_0, y_0, z_0\right) + \left(\frac{\Delta x}{2}\right)\frac{\partial E_y}{\partial x}\Big|_{x_0,y_0,z_0} + \left(\frac{\Delta z}{2}\right)\frac{\partial E_y}{\partial z}\Big|_{x_0,y_0,z_0} + \cdots$$

Similarly for the top face $\left(x+\frac{\Delta x}{2}, y+\Delta y, z+\frac{\Delta z}{2}\right)$ we have:

$$E_y^{\text{top}} = E_y\left(x_0, y_0, z_0\right) + \left(\frac{\Delta x}{2}\right)\frac{\partial E_y}{\partial x}\Big|_{x_0,y_0,z_0} + \left(\frac{\Delta z}{2}\right)\frac{\partial E_y}{\partial z}\Big|_{x_0,y_0,z_0} + \Delta y\frac{\partial E_y}{\partial y}\Big|_{x_0,y_0,z_0} \cdots$$

So the net flux between top and bottom is:

$$\Phi_{\textbf{top-bottom}} = \Delta x\Delta z E_y^{\text{top}} - \Delta x\Delta z E_y^{\text{bottom}}$$

The negative sign arises because the electric field points into one surface (chosen to be the bottom) and out of the other (top). Thus,

$$\Phi_{\text{top-bottom}} = \Delta x \Delta z \begin{bmatrix} E_y(x_0, y_0, z_0) + \left(\frac{\Delta x}{2}\right)\frac{\partial E_y}{\partial x} + \left(\frac{\Delta z}{2}\right)\frac{\partial E_y}{\partial z} + \Delta y \frac{\partial E_y}{\partial y} \\ -E_y(x_0, y_0, z_0) - \left(\frac{\Delta x}{2}\right)\frac{\partial E_y}{\partial x} - \left(\frac{\Delta z}{2}\right)\frac{\partial E_y}{\partial z} \end{bmatrix}$$

$$\Phi_{\text{top-bottom}} = \Delta x \Delta y \Delta z \frac{\partial E_y}{\partial y}$$

We can apply the same procedure to the other two pairs of sides:

$$\Phi_{\text{front-back}} = \Delta x \Delta y \Delta z \frac{\partial E_x}{\partial x}$$

$$\Phi_{\text{left-right}} = \Delta x \Delta y \Delta z \frac{\partial E_z}{\partial z}$$

So the total flux passing through this infinitesimal volume is:

$$\Phi_i = \oint_{S_i} \mathbf{E} \cdot d\mathbf{A} = \Delta x \Delta y \Delta z \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)$$

$$\Phi_i = dV \ \nabla \cdot \mathbf{E}$$

Here we have introduced the volume element $dV = \Delta x \Delta y \Delta z$, and the <u>divergence operator</u>:

$$\nabla \cdot \mathbf{E} \equiv \text{div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Where the <u>gradient operator</u> is defined by:

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}$$

Now we can sum the contributions from all infinitesimal volumes comprising the full volume V:

$$\Phi = \sum_i \Phi_i = \sum_i \oint_{S_i} \mathbf{E} \cdot d\mathbf{A}$$

$$= \sum_i dV_i \left(\nabla \cdot E\right)\Big|_{x_i, y_i, z_i} \rightarrow \int dV \ \nabla \cdot E$$

$$\Rightarrow \boxed{\oint_S \mathbf{E} \cdot d\mathbf{A} = \int_V dV \ \nabla \cdot \mathbf{E}}$$

This forms Gauss' Theorem, or the Divergence Theorem. It states that the surface integral of $\mathbf{E} \cdot d\mathbf{A}$ can be related to the volume integral of $\nabla \cdot \mathbf{E}$.