Lecture 3

Intellectual themes:

- Design of complex systems
- Modeling and controlling physical systems
- Augmenting physical systems with computation
- Building systems that are robust to uncertainty

Intellectual themes are developed in context of a mobile robot

The goal is to convey a distinct perspective about engineering

Models:

- Signals and systems
- Circuits
- Probability
- Planning

Approach: focus on key concepts and pursue in depth

Software Engineering

We will use programming throughout 6.01:

- as an essential tool for engineers and scientists
- to facilitate learning

Programming is intrinsically interesting, exemplifying the two most important themes in 6.01: Abstraction and modularity

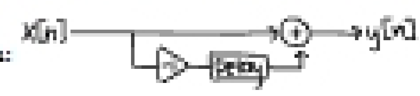
The Signals and Systems Abstraction

Represent a system (physical, mathematical, or computational) by the way it transforms input signal into an output signal

Signal Representation

Difference equation: $y[n] = x[n] - x[n-1]$

Block diagram:

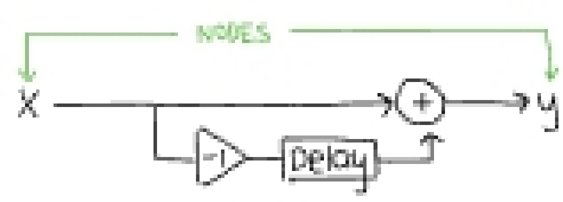


From Samples to Signals

Lumping all of the samples (possibly infinite) into a single object—the signal—simplifies its manipulation

This lumping is analogous to

- Representing coordinates in three-space as points
- Representing lists of numbers as vectors in linear algebra
- Creating an object in Python



Operators manipulate signals rather than individual samples

Nodes represent whole signals (i.e. X and Y)

The boxes operate on those signals

- Delay: shift whole signal to the right one time step
- Add: sum two signals
- Gain: multiply by -1

Signals are primitives

Operators are the means of combination

Operator Notation

Symbols can compactly represent diagrams

Let R represent the right-shift operator:

$Y = RX$ where X represents the whole input signal ($x[n]$ for all n) and Y represents the whole output signal ($y[n]$ for all n)

Systems are concisely represented with operators

The block diagram is the difference machine

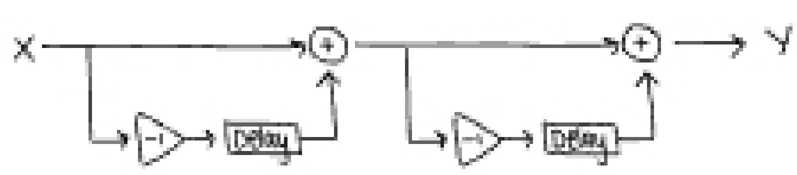
Equivalent representation with R: $Y = X - RX = (1-R)X$

Check yourself: Operator Notation

Correct answer: 2

Let $Y = RX$. Given that equation, we know that $y[n+1] = x[n]$ for all n is true

Operator Representation of a Cascade System



System operations have simggle operator representations

Cascade systems —> multiply operator expressions

Using operator notation

$Y_1 = (1 - R)X$

$Y_2 = (1 - R)Y_1$

Substituting for Y_1:

$Y_2 = (1 - R)(1 - R)X$

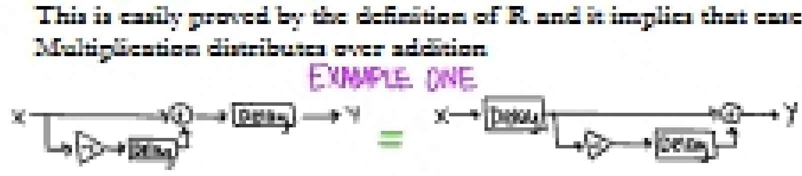
Operator Algebra

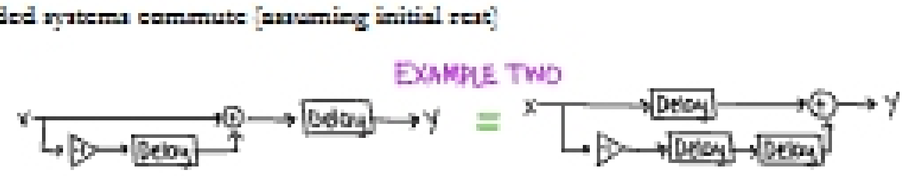
Expressions involving R obey many familiar laws of algebra, such as commutativity

$R(1 - R)X = (1 - R)RX$

This is easily proved by the definition of R and it implies that cascaded systems commute (assuming initial rest)

Multiplication distributes over addition

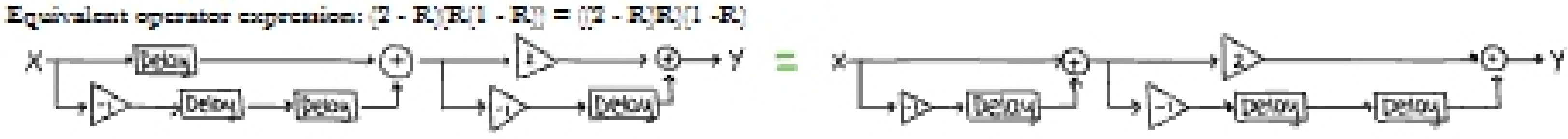




Operator

The associative property similarly holds for operator expressions

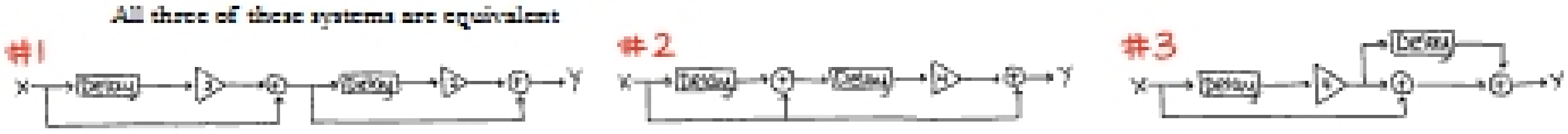
Equivalent operator expression: (2 - R)(R(1 - R)) = ((2 - R)R)(1 -R)



Check Yourself: Operator Algebra

Correct answer: 3

All three of these systems are equivalent

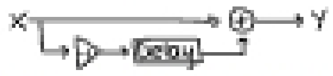


Operator Approach

Applies your existing expertise with polynomials to understand block diagrams, and thereby understand systems

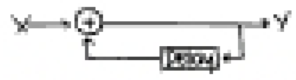
Feedback

Feedback complicates relation between input and output



Y = (1 - R)X

Without feedback, output signal is linear combination of shifted versions of input signal



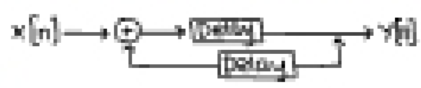
Y = X + RY

Feedback introduces a similar constraint, but now the input signal is a linear combination of shifted versions of the output signal

Check yourself: Feedback

Correct answer: 2

The difference equation that relates X and Y is y[n] = x[n-1] + y[n-2]



Example: Accumulator

To find Y, try step-by-step analysis

Start at rest

The response of the accumulator systems could also be generated by a system with infinitely many paths from input to output, each with one unit of delay more than the previous

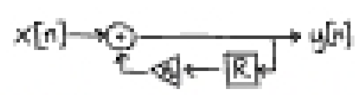
The system functional for the accumulator is the reciprocal of a polynomial R

The product (1 - R) x (1 + R +R² + R³ +...) are reciprocals thus we can write $\frac{Y}{X} = \frac{1}{1-R} = 1 + R + R^2 + R^3 + \ldots$

Geometric Growth

The signal can increase or decrease with time if the loop gain is not 1

Geometric growth gives us the equation $\frac{1}{1-p_0R} = 1 + p_0R + p_0^2R^2 + p_0^3R^3 + \ldots$



This geometric growth arises from cyclic signal flow paths

Cyclic signal flow paths —> persistent responses to transient inputs

These system responses can be characterized by a single number (the pole), which is the base of the geometric sequence

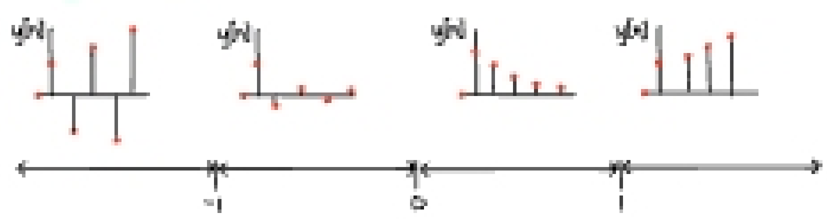
Check yourself: Geometric Growth

Correct answer: 2

What value of p represents the signal? p = -0.5

The value of p determines the rate of growth

- p > 1: magnitude diverges monotonically
- 0 < p < 1: magnitude converges monotonically
- -1 < p <0: magnitude converges, alternating sign
- p < -1: magnitude diverges, alternating sign

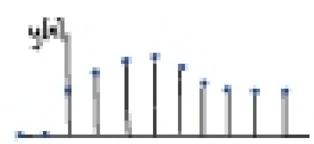


Second-Order Systems

The unit-sample responses of more complicated cyclic systems are more complicated

This response is not geometric

It grows then decays



Second-Order Systems: Equivalent forms

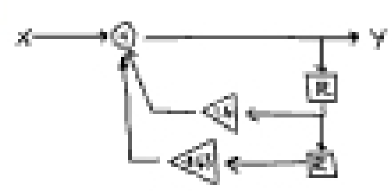
Factor the operator expression to break the system into two similar systems

Y = X + 1.6RY - 0.63R²Y

(1 - 1.6R + 0.63R²)Y = X

(1 - 0.9R)(1 - 0.7R)Y = X

$$\frac{Y}{X} = \frac{1}{1-1.6R-0.63R^2} = \frac{1}{(1-0.9R)(1-0.7R)} = \frac{4.5}{1-0.9R} - \frac{3.5}{1-0.7R}$$



Higher-Order Systems

Systems that can be represented by linear difference equations with constant coefficients have operator representations that are the ratio of polynomials in R

Poles

Poles can be identified by expanding the system functional in partial fractions

The poles are p for 0≤ i ≤ n where n is the order of the n=denominator

Poles can be identified by replacing each R in the system functional with $1/z$

Then the poles are the roots of the denominator polynomial in z

Complex Poles

Powers of complex numbers are easy to compute using polar forms

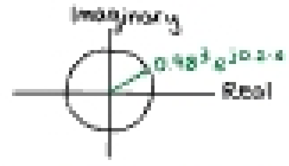
Express the pole at $z = a + jb$ as $re^{j\theta}$ where $r^2 = a^2 + b^2$ and $\tan\theta = b/a$

Then the mode is $(re^{j\theta})^n = r^n e^{j\theta n}$

geometric growth of magnitude

linear growth of angle

Complex Pole Example



Consider a complex pole at re where $r = 0.98$ and $\theta = 0.2$

The nth sample of the corresponding mode is $(re^{j\theta})^n = r^n e^{j\theta n}$

Complex Roots

Difference equations that represent physical systems have real valued coefficients

Difference equations with real-valued coefficients generate real-valued outputs from real-valued inputs