Transfer Function: Laplace Transform of Unit Impulse Response of the System

- Input signal: $u(t) = e^{st}$ where $t \in (-\infty, \infty)$
- Output signal: $y(t) = u(t) * h(t) = \int_0^\infty h(\tau) u(t-\tau) d\tau$

$$= \int_0^\infty h(\tau) e^{s(t-\tau)} d\tau$$

$$= \int_0^\infty h(\tau) e^{st} e^{-s\tau} d\tau$$

$$= \underbrace{\left[\int_0^\infty h(\tau) e^{-s\tau} d\tau\right]}_{=:\mathcal{L}(h) =: H(s)} e^{st}$$

$$=: H(s) e^{st}$$

def: Transfer Function

Laplace transform of the impulse response

- Take $s = jw$

$$u(t) = e^{jwt} = \cos(wt) + j\sin(wt) \rightarrow \boxed{H(jw)} \rightarrow y(t) = H(jw)e^{jwt}$$

Frequency Response

- Input $A\cos wt = \frac{A}{2}(e^{jwt} + e^{-jwt})$
- We know

$$e^{jwt} \rightarrow \boxed{H(s)} \rightarrow H(jw)e^{jwt}$$

$$e^{-jwt} \rightarrow \boxed{H(s)} \rightarrow H(-jw)e^{-jwt}$$

- Complex numbers $H(jw) = Me^{j\phi}, \ H(-jw) = Me^{-j\phi}$

$$y(t) = \frac{A}{2}(H(jw)e^{jwt} + H(-jwt)e^{-jwt})$$

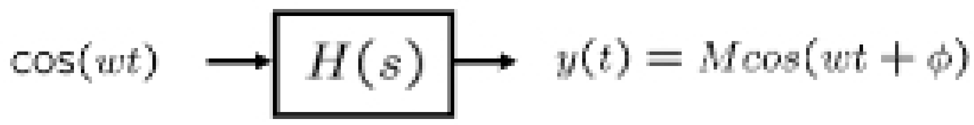
$$= \frac{A}{2}M(e^{j(wt+\phi)} + e^{-j(wt+\phi)})$$

$$= AM\cos(wt+\phi)$$

Magnitude

Phase shift

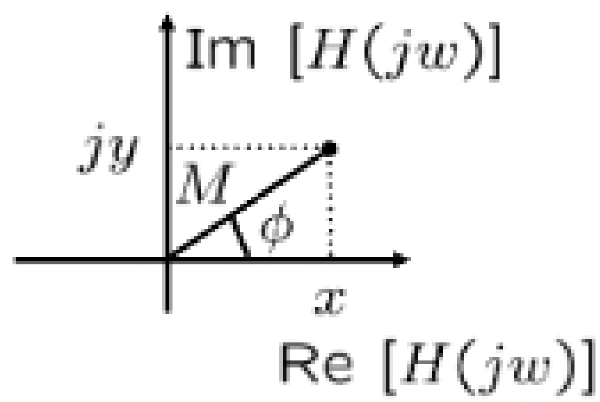
Frequency Response



$$H(jw) = Me^{j\phi} = x + jy$$

$$M = |H(jw)|$$

$$\phi = \tan^{-1}\left[\frac{\text{Im}[H(jw)]}{\text{Re}[H(jw)]}\right]$$



The Laplace Transform (Appendix B)

- Laplace transform converts a calculus problem (the linear differential equation) to an algebra problem
- How to Use it:
  - Take the Laplace transform of a linear differential equation
  - Solve the algebra problem
  - Take the Inverse Laplace transform to obtain the solution to the original differential equation

def: Laplace transform $F(s) := \mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}dt$

def: Inverse Laplace transform

$$f(t) := \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

The Laplace Transform (Appendix B)

- Laplace Transform of a function f(t)

$$F(s) := \mathcal{L}f(t) = \int_0^\infty f(t)e^{-st}dt$$

- Convolution integral

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)d\tau$$
$$= \int_0^t f_2(\tau)f_1(t-\tau)d\tau$$

$$\mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)$$

Properties of Laplace Transforms (page 641-643)

- Linearity

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$
$$= aF(s) + bG(s)$$

- Time Delay

$$\mathcal{L}[f(t-\lambda)] = \int_0^\infty f(t-\lambda)e^{-st}dt$$
$$= \int_0^\infty f(\tau)e^{-s(\tau+\lambda)}d\tau$$
$$= e^{-s\lambda}\mathcal{L}[f(t)]$$

Non-rational function