6.003: Signals and Systems

Laplace and Z Transforms

October 1, 2009

Mid-term Examination #1

Wednesday, October 7, 7:30-9:30pm, Walker Memorial.

No recitations on the day of the exam.

Coverage: DT Signals and Systems
Lectures 1–5
Homeworks 1–4

Homework 4 will include practice problems for mid-term 1. However, it will not collected or graded. Solutions will be posted.

Closed book: 1 page of notes ($8\frac{1}{2} \times 11$ inches; front and back).

Designed as 1-hour exam; two hours to complete.

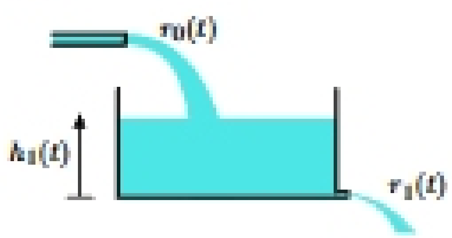
Review sessions during open office hours.

Conflict? Contact freeman@mit.edu before Friday, October 2, 5pm.

Last Time

Many continuous-time systems can be represented with differential equations.

Example: leaky tank



Differential equation representation:

$$\tau \dot{r}_1(t) = r_0(t) - r_1(t)$$

Last time we considered two methods to solve differential equations:

- solving homogeneous and particular equations
- singularity matching

Solving Differential Equations with Laplace Transform

The Laplace transform provides a particularly powerful method of solving differential equations — it transforms a differential equation into an algebraic equation.

Method (where $\mathcal{L}$ represents the Laplace transform):

differential equation $\xrightarrow{\mathcal{L}}$ algebraic equation

↓ solve

algebraic answer $\xrightarrow{\mathcal{L}^{-1}}$ solution to differential equation

Laplace Transform: Definition

Laplace transform maps a function of time t to a function of s.

$$X(s) = \int x(t)e^{-st}dt$$

There are two important variants:

Unilateral (18.03)

$$X(s) = \int_0^{\infty} x(t)e^{-st}dt$$

Bilateral (6.003)

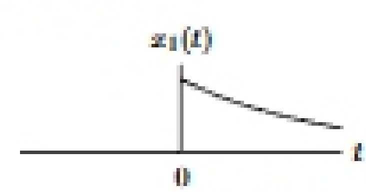
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Both share important properties — will discuss differences later.

Laplace Transforms

Example: Find the Laplace transform of $x_1(t)$:

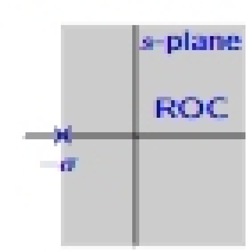
$$x_1(t) = \begin{cases} Ae^{-\sigma t} & \text{if } t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_1(s) = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt = \int_0^{\infty} Ae^{-\sigma t}e^{-st}dt = \left.\frac{Ae^{-(s+\sigma)t}}{-(s+\sigma)}\right|_0^{\infty} = \frac{A}{s+\sigma}$$

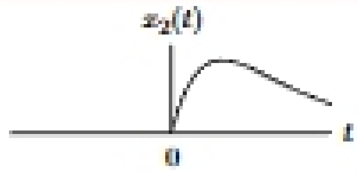
provided $\text{Re}\{s+\sigma\} > 0$ which implies that $\text{Re}\{s\} > -\sigma$.

$$\frac{A}{s+\sigma} \; ; \quad \text{Re}\{s\} > -\sigma$$



Check Yourself

$$x_2(t) = \begin{cases} e^{-t} - e^{-2t} & \text{if } t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



Which of the following is the Laplace transform of $x_2(t)$?

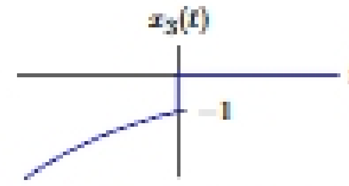
1. $X_2(s) = \frac{1}{(s+1)(s+2)}$; $\text{Re}\{s\} > -1$
2. $X_2(s) = \frac{1}{(s+1)(s+2)}$; $\text{Re}\{s\} > -2$
3. $X_2(s) = \frac{s}{(s+1)(s+2)}$; $\text{Re}\{s\} > -1$
4. $X_2(s) = \frac{s}{(s+1)(s+2)}$; $\text{Re}\{s\} > -2$
5. none of the above

Regions of Convergence

Left-sided signals have left-sided Laplace transforms (bilateral only).

Example:

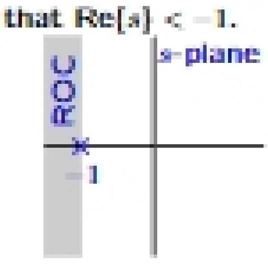
$$x_3(t) = \begin{cases} -e^{-t} & \text{if } t \le 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_3(s) = \int_{-\infty}^{\infty} x_3(t)e^{-st}dt = \int_{-\infty}^{0} -e^{-t}e^{-st}dt = \frac{-e^{-(s+1)t}}{-(s+1)}\Bigg|_{-\infty}^{0} = \frac{1}{s+1}$$

provided $\text{Re}\{s+1\} < 0$ which implies that $\text{Re}\{s\} < -1$.

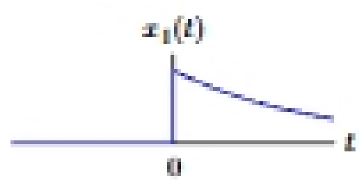
$$\frac{1}{s+1} \;;\quad \text{Re}\{s\} < -1$$



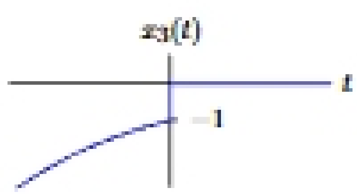
Left- and Right-Sided Signals

We can concisely express left- and right-sided signals by multiplication with step functions.

$$x_1(t) = \begin{cases} e^{-t} & \text{if } t \ge 0 \\ 0 & \text{otherwise} \end{cases} = e^{-t}u(t)$$



$$x_3(t) = \begin{cases} -e^{-t} & \text{if } t \le 0 \\ 0 & \text{otherwise} \end{cases} = -e^{-t}u(-t)$$

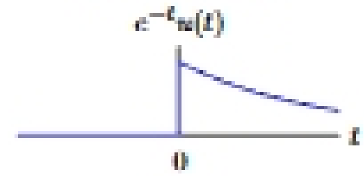


Left- and Right-Sided ROCs

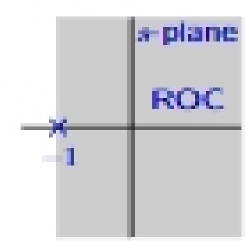
Laplace transforms of left- and right-sided exponentials have the same form (except −); with left- and right-sided ROCs, respectively.

time function Laplace transform

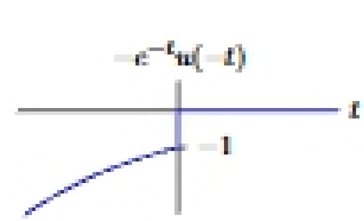
$e^{-t}u(t)$



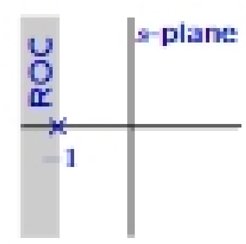
$$\frac{1}{s+1}$$



$-e^{-t}u(-t)$



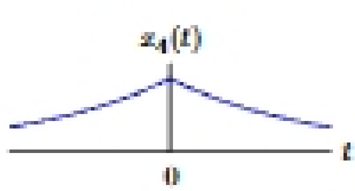
$$\frac{1}{s+1}$$



Check Yourself

Find the Laplace transform of $x_4(t)$.

$$x_4(t) = e^{-|t|}$$



1. $X_4(s) = \frac{2}{1-s^2}$; $-\infty < \text{Re}\{s\} < \infty$
2. $X_4(s) = \frac{2}{1-s^2}$; $-1 < \text{Re}\{s\} < 1$
3. $X_4(s) = \frac{2}{1+s^2}$; $-\infty < \text{Re}\{s\} < \infty$
4. $X_4(s) = \frac{2}{1+s^2}$; $-1 < \text{Re}\{s\} < 1$
5. none of the above

Solving Differential Equations with Laplace Transforms

Solve the following differential equation:

$$\dot{y}(t) + y(t) = \delta(t)$$

Take the Laplace transform of this equation.

$$\mathcal{L}\{\dot{y}(t) + y(t)\} = \mathcal{L}\{\delta(t)\}$$

The Laplace transform of a sum is the sum of the Laplace transforms (prove this as an exercise).

$$\mathcal{L}\{\dot{y}(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t)\}$$

What's the Laplace transform of a derivative?

Laplace transform of a derivative

Assume that $X(s)$ is the Laplace transform of $x(t)$:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Find the Laplace transform of $y(t) = \dot{x}(t)$.

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt$$
$$= \int_{-\infty}^{\infty} \dot{x}(t)e^{-st}dt$$
$$= x(t)e^{-st}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)(-se^{-st})dt$$

The first term must be zero since $X(s)$ converged. Thus

$$Y(s) = s\int_{-\infty}^{\infty} x(t)e^{-st}dt = sX(s)$$

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$\mathcal{L}\{\dot{y}(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t)\}$$

Let $Y(s)$ represent the Laplace transform of $y(t)$.

Then $sY(s)$ is the Laplace transform of $\dot{y}(t)$.

$$sY(s) + Y(s) = \mathcal{L}\{\delta(t)\}$$

What's the Laplace transform of the impulse function?

Laplace transform of the impulse function

Let $x(t) = \delta(t)$.

$$X(s) = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt$$
$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\big|_{t=0}\, dt$$
$$= \int_{-\infty}^{\infty} \delta(t)\ 1\ dt$$
$$= 1$$

Sifting property: $\delta(t)$ **sifts** out the value of the integrand at $t = 0$.

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$sY(s) + Y(s) = \mathcal{L}\{\delta(t)\} = 1$$

This is a simple algebraic expression. Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

We've seen this Laplace transform previously.

$y(t) = e^{-t}u(t)$ (why not $y(t) = -e^{-t}u(-t)$?)

Notice that we solved the differential equation $\dot{y}(t) + y(t) = \delta(t)$ without computing homogeneous and particular solutions and without singularity matching.

Solving Differential Equations with Laplace Transforms

Summary of method.

Start with differential equation:

$$\dot{y}(t) + y(t) = \delta(t)$$

Take the Laplace transform of this equation:

$$sY(s) + Y(s) = 1$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

Take inverse Laplace transform (by recognizing form of transform):

$$y(t) = e^{-t}u(t)$$

Solving Differential Equations with Laplace Transforms

Recognizing the form ...

Is there a more systematic way to take an inverse Laplace transform?

Yes ... and no.

Formally,

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

but this integral is not generally easy to compute.

This equation can be useful to prove theorems.

We will find better ways (e.g., partial fractions) to compute inverse transforms for common systems.