Chapter 6 Differential Analysis of Fluid Flow

Inviscid flow: Euler's equations of motion

Flow fields in which the shearing stresses are zero are said to be inviscid, nonviscous, or frictionless. for fluids in which there are no shearing stresses the normal stress at a point is independent of direction:

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$

For an inviscid flow in which all the shearing stresses are zero, and the normal stresses are replaced by −p, the Navier-Stokes Equations reduce to Euler's equations

$$\rho g - \nabla p = \rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right]$$

In Cartesian coordinates:

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

The Bernoulli equation derived from Euler's equations

The Bernoulli equation can also be derived, starting from Euler's equations. For inviscid, incompressible fluids, we end up with the same equation

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = const$$

It is often convenient to write the Bernoulli equation between two points (1) and (2) along a streamline and to express the equation in the "head" form by dividing each term by g so that

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

The Bernoulli equation is restricted to the following:

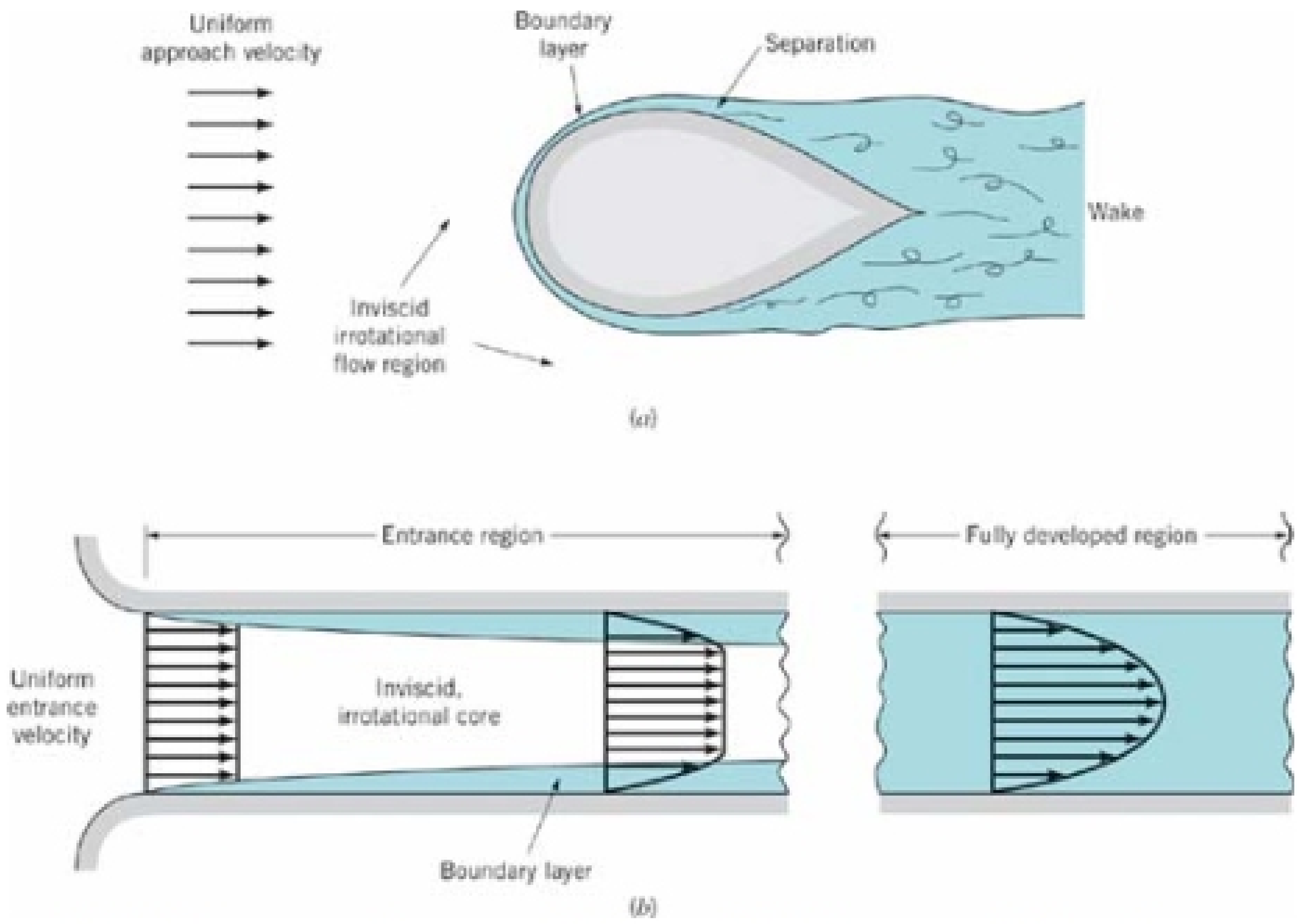
- inviscid flow
- steady flow
- incompressible flow
- flow along a streamline

<u>The Irrotational Flow and corresponding Bernoulli equation</u>

If we make one additional assumption—that the flow is irrotational $\nabla\times\mathbf{V}=0$ —the analysis of inviscid flow problems is further simplified. The Bernoulli equation has exactly the same form at that for inviscid flows:

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

but it can now be applied between any two points in the flow field, not limited to applications along a streamline.



Various regions of flow: (a) around bodies;
(b) through channels

The Velocity Potential

For an irrotational flow:

$$\nabla \times \mathbf{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathbf{k} = 0$$

So we have

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

It follows that in this case the velocity components can be expressed in terms of a scalar function ϕ(x, y, z, t), called velocity potential, as

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}$$

In vector form:

$$\mathbf{V} = \nabla \phi$$