Recursion

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Notes

Recursive Algorithms

A *recursive algorithm* is one in which objects are defined in terms of other objects of the same type.

Advantages:

- Simplicity of code
- Easy to understand

Disadvantages:

- Memory
- Speed
- Possibly redundant work

Tail recursion offers a solution to the memory problem, but really, do we *need* recursion?

Notes

Recursive Algorithms
Analysis

We've already seen how to analyze the running time of algorithms. However, to analyze recursive algorithms, we require more sophisticated techniques.

Specifically, we study how to define & solve *recurrence relations*.

Notes

Motivating Example

Factorial

Recall the factorial function.

$$n! = \begin{cases} 1 & \text{if } n = 1 \\ n \cdot (n-1)! & \text{if } n > 1 \end{cases}$$

Consider the following (recursive) algorithm for computing $n!$:

Algorithm (FACTORIAL)

```
INPUT      : n ∈ N
OUTPUT     : n!
if n = 1 then
    return 1
end
else
    return FACTORIAL(n − 1) × n
end
```

Notes

Motivating Example

Factorial - Analysis?

How many multiplications $M(n)$ does FACTORIAL perform?

- When $n = 1$ we don't perform any.
- Otherwise we perform 1.
- *Plus* how ever many multiplications we perform in the recursive call, FACTORIAL$(n-1)$.
- This can be expressed as a formula (similar to the definition of $n!$.

$$\begin{aligned} M(0) &= 0 \\ M(n) &= 1 + M(n-1) \end{aligned}$$

- This is known as a *recurrence relation*.

Notes

Recurrence Relations I

Definition

Definition

A *recurrence relation* for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms in the sequence,

$$a_0, a_1, \ldots, a_{n-1}$$

for all integers $n \geq n_0$ where n_0 is a nonnegative integer.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Notes

Recurrence Relations II

Definition

Consider the recurrence relation: $a_n = 2a_{n-1} - a_{n-2}$.
It has the following sequences a_n as solutions:

1. $a_n = 3n$,
2. $a_n = n + 1$, and
3. $a_n = 5$.

Initial conditions + recurrence relation uniquely determine the sequence.

Notes

Recurrence Relations III

Definition

Example

The Fibonacci numbers are defined by the recurrence,

$$\begin{aligned} F(n) &= F(n-1) + F(n-2) \\ F(1) &= 1 \\ F(0) &= 1 \end{aligned}$$

The solution to the Fibonacci recurrence is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

(your book derives this solution).

Notes

Recurrence Relations IV

Definition

More generally, recurrences can have the form

$$T(n) = \alpha T(n - \beta) + f(n), \quad T(\delta) = c$$

or

$$T(n) = \alpha T\left(\frac{n}{\beta}\right) + f(n), \quad T(\delta) = c$$

Note that it may be necessary to define *several* $T(\delta)$, *initial conditions*.

Notes