

Lecture Outline (Related Rates)

Wednesday, March 5

Announcements

1. Final is on Monday, 3/17/08 at 8:30 AM. If you can't make it then, the only other time you can take it is at the University scheduled make-up time which is Monday night from 7-10 (but you have a very good excuse to take it at this time.)
2. The last day of class (3/14/08) will be a review session for the final - come prepared with questions.

Motivation for Related Rates

As we saw last time, there are many applications of implicit differentiation. Today and tomorrow we are going to use it to talk about very concrete applications called related rates.

To illustrate the idea of related rates, consider a balloon (we'll assume it is shaped like a sphere) being filled with air. Let's consider two quantities: the volume of the balloon and the radius of the balloon. Of course, we know how they are related: $V = 4/3\pi r^3$ (the volume of a sphere). Except that in this problem, we have another, hidden variable: the time t . Because as time progresses, both the radius and the volume are functions of time.

If we are using a machine to pump air into the balloon, there is a good chance we know the rate at which the air is being pumped in. The rate at which the air is being pumped in is precisely the rate of change of the volume (with respect to time) which is $\frac{dV}{dt}$. But what about the radius? At what rate is the radius changing? That is, what is $\frac{dr}{dt}$? The trick is to look at the equation $V = \frac{4}{3}\pi r^3$ and implicitly differentiate *with respect to t !* We get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Suppose the problem said that the air was being pumped in at $10 \text{ cm}^3/\text{sec}$ and asked for the rate at which the radius was changing when the radius was 10 cm. Then

$$\begin{aligned} 10 &= 4\pi(10)^2 \frac{dr}{dt} \\ \frac{dr}{dt} &= \frac{1}{40\pi} \end{aligned}$$

So the radius is changing at $\frac{1}{40\pi} \text{ cm/sec}$ when the radius is 10 cm.

Theory: Related Rates

The strategy for solving related rates is:

1. Read the problem carefully.
2. Introduce notation. Assign variables to all quantities that are functions of time.
3. Draw a picture if possible.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (see Example 3 in Section 4.1.)
6. Use the chain rule to (implicitly) differentiate both sides of the equation *with respect to* t .
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example: A Related Rate Problem

Problem: A plane flying horizontally at an altitude of 1 mile and a speed of 500 miles/hour passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.

Discussion: After reading the problem carefully, the next few steps in a related rates problem are to draw a diagram, introduce variables, write down what we know in terms of the variables, and set up an equation relating the variables. As we introduce variables, we'll go back to the diagram and fill it out more completely.

The key point here is that there are three different types of quantities in any related rates problem. Namely:

1. Quantities that do not change at all (ones that are the same no matter what the time.)
2. Quantities that vary depending on time.
3. Quantities that are only true at a specific moment in time.

Solution: Consider the above problem. First we are told that the plane has an altitude of 1 mile. That is true no matter what the time, so that is a quantity of type 1. Secondly, we are told the plane has a speed of 500 miles per hour; this is also true no matter what the time, and is a quantity of type 1. The problem mentions the distance from the plane to the station. This changes depending on the time, so it is a quantity of type 2. Finally the problem mentions the plane being 2 miles away. This is only true at a specific moment in

time, and hence is of type 3.

So what do we do with different types of quantities? First you should assign a variable to every quantity of type 2; after all, each such quantity varies depending on time, and should be a variable. In our problem, the distance to the station is such a quantity: let's call it d .

Secondly, we can label the diagram with the quantities of type 1 and 2 (we want our picture to be true no matter what time it is, so we don't use any quantities of type 3). In our problem, we label altitude of the plane as 1 mile, the distance from the plane to the station as d , and I would draw a little arrow in the direction the plane is going and label it 500 miles/hour.

Also, if some quantity of type 1 is a rate of change, it should be the derivative of a variable (a quantity of type 2). Now the speed of the plane is how fast the horizontal distance between the station and the plane is changing. We don't have a variable for that, but we now see that we should. Let's call it x and label it in our diagram (it's a quantity of type 2). Then we know that $\frac{dx}{dt} = 500$.

Finally, before setting up any equations, let's notice that what we are trying to find is $\frac{dd}{dt}$ when the plane is 2 miles away.

Looking back at the diagram, we realize that we have a right triangle, and can write:

$$x^2 + 1 = d^2.$$

This equation is true, but it's not immediately helpful since what we really need to find is $\frac{dd}{dt}$, and so we need an equation that involves $\frac{dd}{dt}$. We can get such an equation by differentiating the equation above with respect to t .

$$\begin{aligned} 2x\left(\frac{dx}{dt}\right) &= 2d\left(\frac{dd}{dt}\right) \\ \frac{dd}{dt} &= \frac{x}{d}\left(\frac{dx}{dt}\right) \end{aligned}$$

Whoop! Now we know that $\frac{dd}{dt}$ is always equal to $\frac{x}{d}\left(\frac{dx}{dt}\right)$. We know that $\frac{dx}{dt}$ is always 500, so let's substitute that in to find that $\frac{dd}{dt}$ is always equal to $500x/d$.

If we just wanted to know what $\frac{dd}{dt}$ was in general, this would be the answer. But we want to know what $\frac{dd}{dt}$ is at a specific moment in time, namely when the plane is 2 miles away from the station. This is when our quantities of type 3 come into play. At that specific moment in time, we know that $d = 2$. So $\frac{dd}{dt} = 250x$. But we want a numerical answer, so we'd better figure out what x is when the plane is 2 miles away from the station. Fortunately, we know that $d^2 = x^2 + 1$, so if $d = 2$, we can solve to find that $x = \sqrt{3}$. Plugging that in, we finally find that $\frac{dd}{dt} = 250\sqrt{3}$ when the plane is 2 miles away from the station. That is, when the plane is 2 miles away from the station, the distance between the plane and the station is changing (increasing) at $250\sqrt{3}$ which is about 433 miles per hour.