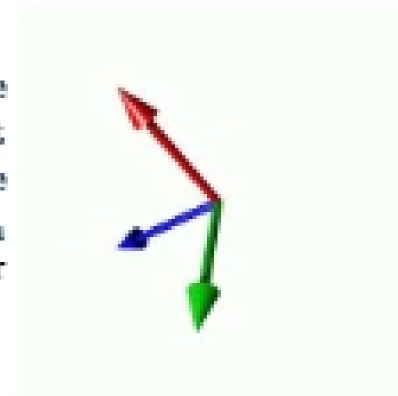
BASIS — Math 21b, O. Knill

LINEAR SUBSPACE. A subset X of $\mathbf{R}^n$ which is closed under addition and scalar multiplication is called a **linear subspace** of $\mathbf{R}^n$.

WHICH OF THE FOLLOWING SETS ARE LINEAR SPACES?

a) The kernel of a linear map.
b) The image of a linear map.
c) The upper half plane.
d) the line $x + y = 0$.
e) The plane $x + y + z = 1$.
f) The unit circle.

BASIS. A set of vectors $\vec{v}_1, \dots, \vec{v}_m$ is a **basis** of a linear subspace X of $\mathbf{R}^n$ if they are **linear independent** and if they **span** the space X. Linear independent means that there are no nontrivial **linear relations** $a_1\vec{v}_1 + \dots + a_m\vec{v}_m = 0$. Spanning the space means that very vector $\vec{v}$ can be written as a linear combination $\vec{v} = a_1\vec{v}_1 + \dots + a_m\vec{v}_m$ of basis vectors. A **linear subspace** is a set containing $\{\vec{0}\}$ which is closed under addition and scaling.



EXAMPLE 1) The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis in the three dimensional space. If $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$, then $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$ and this representation is unique. We can find the coefficients by solving $A\vec{x} = \vec{v}$, where A has the v_i as column vectors. In our case, $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$ had the unique solution $x = 1, y = 2, z = 3$ leading to $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$.

EXAMPLE 2) Two nonzero vectors in the plane which are not parallel form a basis.
EXAMPLE 3) Three vectors in $\mathbf{R}^3$ which are in a plane form **not a basis**.
EXAMPLE 4) Two vectors in $\mathbf{R}^3$ do **not** form a basis.
EXAMPLE 5) Three nonzero vectors in $\mathbf{R}^3$ which are not contained in a single plane form a basis in $\mathbf{R}^3$.
EXAMPLE 6) The columns of an invertible $n \times n$ matrix form a basis in $\mathbf{R}^n$ as we will see.

FACT. If $\vec{v}_1, ..., \vec{v}_n$ is a basis, then every vector $\vec{v}$ can be represented **uniquely** as a linear combination of the $\vec{v}_i$. $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.

REASON. There is at least one representation because the vectors $\vec{v}_i$ span the space. If there were two different representations $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$ and $\vec{v} = b_1\vec{v}_1 + \dots + b_n\vec{v}_n$, then subtraction would lead to $0 = (a_1 - b_1)\vec{v}_1 + ... + (a_n - b_n)\vec{v}_n$. This nontrivial linear relation of the v_i is forbidden by assumption.

FACT. If n vectors $\vec{v}_1, ..., \vec{v}_n$ span a space and $\vec{w}_1, ..., \vec{w}_m$ are linear independent, then $m \leq n$.

REASON. This is intuitively clear in dimensions up to 3. You can not have more then 4 vectors in space which are linearly independent. We will give a precise reason later.

A BASIS DEFINES AN INVERTIBLE MATRIX. The $n \times n$ matrix $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}$ is invertible if and only if $\vec{v}_1, \dots, \vec{v}_n$ define a basis in $\mathbf{R}^n$.

EXAMPLE. In the example 1), the 3×3 matrix A is invertible.

FINDING A BASIS FOR THE KERNEL. To solve $Ax = 0$, we bring the matrix A into the reduced row echelon form rref(A). For every non-leading entry in rref(A), we will get a free variable t_i. Writing the system $Ax = 0$ with these free variables gives us an equation $\vec{x} = \sum_i t_i\vec{v}_i$. The vectors $\vec{v}_i$ form a basis of the kernel of A.

REMARK. The problem to find a basis for all vectors $\vec{w}_i$ which are orthogonal to a given set of vectors, is equivalent to the problem to find a basis for the kernel of the matrix which has the vectors $\vec{w}_i$ in its rows.

FINDING A BASIS FOR THE IMAGE. Bring the $m \times n$ matrix A into the form rref(A). Call a column a **pivot column**, if it contains a leading 1. The corresponding set of column vectors of the original matrix A form a basis for the image because they are linearly independent and are in the image. Assume there are k of them. They span the image because there are $(k-n)$ non-leading entries in the matrix.

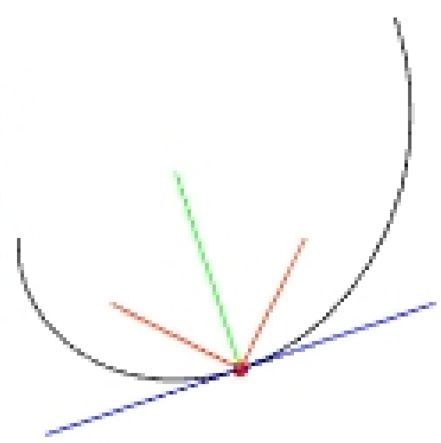
REMARK. The problem to find a basis of the subspace generated by $\vec{v}_1, \dots, \vec{v}_n$, is the problem to find a basis for the image of the matrix A with column vectors $\vec{v}_1, ..., \vec{v}_n$.

EXAMPLES.
1) Two vectors on a line are linear dependent. One is a multiple of the other.
2) Three vectors in the plane are linear dependent. One can find a relation $a\vec{v}_1 + b\vec{v}_2 = \vec{v}_3$ by changing the size of the lengths of the vectors $\vec{v}_1, \vec{v}_2$ until $\vec{v}_3$ becomes the diagonal of the parallelogram spanned by $\vec{v}_1, \vec{v}_2$.
3) Four vectors in three dimensional space are linearly dependent. As in the plane one can change the length of the vectors to make $\vec{v}_4$ a diagonal of the parallelepiped spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

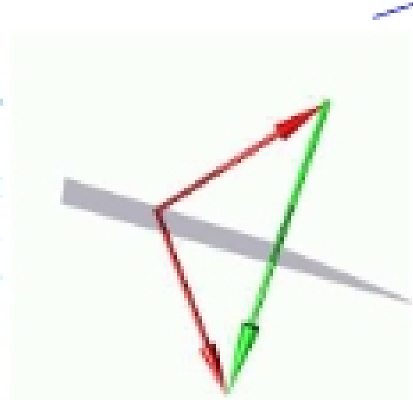
EXAMPLE. Let A be the matrix $A = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$. In reduced row echelon form is $B = \text{rref}(A) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$.
To determine a basis of the kernel we write $Bx = 0$ as a system of linear equations: $x + y = 0, z = 0$. The variable y is the free variable. With $y = t$, $x = -t$ is fixed. The linear system rref$(A)x = 0$ is solved by $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. So, $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis of the kernel.

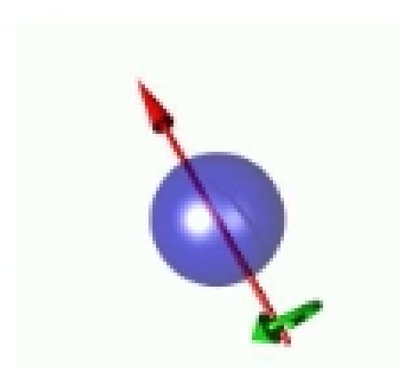
EXAMPLE. Because the first and third vectors in rref(A) are columns with leading 1's, the first and third columns $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ of A form a basis of the image of A.

WHY DO WE INTRODUCE BASIS VECTORS? Wouldn't it be just easier to look at the standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$ only? The reason for more general basis vectors is that they allow **a more flexible adaptation** to the situation. A person in Paris prefers a different set of basis vectors than a person in Boston. We will also see that in many applications, problems can be solved easier with the right basis.



For example, to describe the reflection of a ray at a plane or at a curve, it is preferable to use basis vectors which are tangent or orthogonal. When looking at a rotation, it is good to have one basis vector in the axis of rotation, the other two orthogonal to the axis. Choosing the right basis will be especially important when studying differential equations.





A PROBLEM. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Find a basis for ker$(A)$ and im(A).

SOLUTION. From rref$(A) = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix}$ we see that $= \vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is in the kernel. The two column vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \left[\ 2, 1, 1\ \right]$ of A form a basis of the image because the first and third column are pivot columns.