Worksheet #2: Vector-Valued Functions

I. Lines and Curves in Space

1. (Dimension matters!)

   I. In the xy-plane, the set of points described by the equation $y = x$ is a:

   LINE PLANE

   II. In the xyz-plane, the set of points described by the equation $y = x$ is a:

   LINE PLANE

   III. Suppose f and g are functions. Explain why the set of points in the xyz-plane described by a relationship $y = f(x)$ is a surface. Is the set of points described by $z = g(x, y)$ a curve or a surface?

2. Let l be the line parallel to $\vec{v} = \hat{\imath} + \hat{\jmath} - 3\hat{k}$ that passes through $(1, 0, 1)$.

   I. Find a parametric description of l.

   II. Determine if l lies on the plane $x + 2y + z = 2$. Explain your answer!

3. Let l be the line that passes through $(1, 2, -1)$ and $(3, 4, 0)$.

   I. Find a parametric description of l.

   II. Find a value for a so the line l lies on the plane $x - y + az = 3$ or explain why there is no such value for a.

4. Find a parametric description of the line passing through $(-1, 2, 3)$ that is perpendicular to both $\vec{u} = \hat{\imath} + \hat{k}$ and $\vec{v} = \hat{\jmath} - \hat{k}$.

5. Find a vector parallel to the line $\vec{r}(t) = \langle 3t + 2, 4 - t, 6t + 5\rangle$.

6. The curve C is described by the vector-valued function $\vec{r}(t) = \langle t^2, 2t + 1, 3t^2 + 4t\rangle$

   I. Determine whether the point $(x, y, z) = (1, 3, 4)$ lies on the curve C. Explain your answer.

   II. Find all points (x, y, z) for which which the curve intersects the surface $4x - y^2 + z = 28$.

II. Calculus of Vector-Valued Functions

7. Consider the vector-valued function $\vec{r}(t) = \left\langle \sqrt{t+4}, \dfrac{2}{t^2 - 4}, \ln(2t - 6) \right\rangle$

   I. Find the domain of $\vec{r}(t)$.

   II. Determine the intervals on which $\vec{r}(t)$ is continuous.

8. Consider the vector-valued function $\vec{r}(t) = \left\langle 5t, \dfrac{e^{2t} - 1}{t}, t^3 - 4t \right\rangle$

I. Find the domain of $\vec{r}(t)$.

II. Is $\vec{r}(t)$ continuous at $t = 0$? Explain your answer.

III. Define a new vector-valued function $\vec{R}(t)$ by requiring that $\vec{R}(t) = \vec{r}(t)$ when $t \neq 0$ and $\vec{R}(t) = \langle 0, a, 0\rangle$ when $t = 0$. Determine a value for a so the function $\vec{R}(t)$ is continuous at $t = 0$ or explain why no such value for a exists.

9. Find $\vec{r}\,'(t)$ for the following curves:

I. $\vec{r}(t) = \langle 1 + 2t, 3 - 4t, 6 + 7t\rangle$

II. $\vec{r}(t) = \langle te^t, \ln(1 + t^2), 6e^{2t}\rangle$

III. $\vec{r}(t) = \langle t^2, \cos(2t), \sin(t^3)\rangle$

10. A curve C is described by the vector-valued function $\vec{r}(t) = \langle 1 + 2t, 3 - 4t, 6 + 7t\rangle$.

I. Find a vector parallel to the tangent line to the curve when $t = 1$.

II. Find all t-values for which $\vec{r}(t)$ and $\vec{r}\,'(t)$ are orthogonal.

III. Find the unit tangent vector $\hat{T}(t)$ to the curve.

11. A curve C is described by the vector-valued function $\vec{r}(t) = \langle t^2, 4 - 2t^2, t^2 + 2\rangle$.

I. Find all t-values for which $\vec{r}(t)$ and $\vec{r}\,'(t)$ are orthogonal.

II. Find the unit tangent vector $\hat{T}(t)$ to the curve for $t \geq 0$.

III. Find a parametric description of the tangent line to the curve when $t = 2$.

12. Consider the cubic $y = x^3$.

I. Find the Cartesian equation for the tangent line to the curve when $y = 8$.

II. A parameterization $\vec{r}(t)$ for the cubic is $\vec{r}(t) = \langle t, t^3\rangle$. Find a parametric description of the tangent line when $y = 8$.

III. Do your answers to Parts I. and II. describe the same line? Explain!

13. Suppose that C is a curve described by the differentiable vector-valued function $\vec{r}(t)$ and assume additionally that $|\vec{r}(t)| = 4$ for all t.

I. Is it true that $\vec{r}\,'(t) = \vec{0}$ for all t? Explain!

II. Is it true that $\vec{r}\,'(t)$ is constant for all time t? Explain!

III. Find an explicit example of a vector-valued function $\vec{r}(t)$ for which $|\vec{r}(t)| = 4$ but for which $|\vec{r}\,'(t)| = t$ for all time t.

IV. Show that $\vec{r}(t)$ and $\vec{r}\,'(t)$ are orthogonal for all time t.

14. The starting position of a particle is given by $\vec{r}(0) = \langle 0, 0, 0\rangle$ and its acceleration is described by the vector-valued function $\vec{a}(t) = \langle -3, 0, -6t\rangle$ for $t \geq 0$. Suppose the initial velocity is given by $\vec{v}(0) = \langle 60, 10, 75\rangle$. Find:

I. The velocity function.

II. The speed function.

III. The position function.

IV. The maximum z-height of the particle when $t \geq 0$.

V. The total distance the particle travels from its starting location once it hits the ground.

15. Given that $\vec{u}(t) = \langle t^2, 3t, 1 \rangle$ and $\vec{v}(t) = \langle 6t, 4t^2, e^t \rangle$, find:

I. $\vec{u}(t) \cdot \vec{v}(t)$

II. $\dfrac{d}{dt}\left[t^2\vec{u}(t)\right]$

III. $\dfrac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right]$

IV. The orthogonal projection of $\vec{v}(t)$ onto $\vec{u}(0)$

V. $\dfrac{d}{dt}\left[\vec{u}(1) \cdot \vec{v}(t)\right]$

VI. $\dfrac{d}{dt}\left[proj_{\vec{u}(0)}\,\vec{v}(t)\right]$ when $t = 3$.

III. Arclength

16. Suppose that a curve C is described by a differentiable vector-valued function $\vec{r}(t)$. Explain how to determine whether the curve is parameterized by arclength.

17. Conceptually, what is the difference between using a (time) parameter t versus the arclength parameter s to describe a curve?

18. Given that $\vec{r}(t)$ is a differentiable vector-valued function that is not parameterized by arclength, explain how to find a description of the curve that uses arclength as a parameter.

19. Given the following vector-valued functions defined on $[a, b]$, find the length of the curve they describe from $t = a$ to $t = b$.

I. $\vec{r}(t) = \langle 2\cos(3t), -2\sin(3t) \rangle$, $[a, b] = [0, \pi]$

II. $\vec{r}(t) = \langle t^3, t^2, t^2 \rangle$, $[a, b] = [0, 1]$

III. $\vec{r}(t) = \left\langle 5t, 4t^{3/2}, -t \right\rangle$, $[a, b] = [0, 4]$

20. For each of the vector-valued functions above, determine whether the curve is parameterized by arclength. For each that is not, give a description that uses arclength as a parameter and give the domain of the arclength parameter!

21. A curve C is described by a differentiable vector-valued function:

$$\vec{r}(t) = \left\langle 4t^{3/2}, 1, \frac{16}{3}t^{3/2} \right\rangle \text{ for } 0 \leq t \leq 4.$$

I. Determine whether the curve is parameterized by arclength. If it is not, give a description of the curve that uses arclength as a parameter. Do not forget to give the domain of the arclength parameter!

II. Find the length of this curve.