

Application 7.4

Earthquake-Induced Vibrations of Multistory Buildings

In this application you are to investigate the response to transverse earthquake ground oscillations of a seven-story building like the one illustrated in Fig. 7.4.15 of the text. Suppose that each of the seven [aboveground] floors weighs 16 tons, so the mass of each is $m = 1000$ (slugs). Also assume a horizontal restoring force of $k = 5$ (tons per foot) between adjacent floors. That is, the internal forces in response to horizontal displacements of the individual floors are those shown in Fig. 7.4.16. It follows that the free transverse oscillations indicated in Fig. 7.4.15 satisfy the equation $\mathbf{M}\mathbf{x}'' = \mathbf{K}\mathbf{x}$ with $n = 7$ and $m_i = 1000$, $k_i = 10,000$ (lb/ft) for $1 \leq i \leq 7$. The system then reduces to the form $\mathbf{x}'' = \mathbf{A}\mathbf{x}$ with

$$\mathbf{A} = \begin{bmatrix} -20 & 10 & 0 & 0 & 0 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & -20 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & -20 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & -20 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 & -20 & 10 \\ 0 & 0 & 0 & 0 & 0 & 10 & -10 \end{bmatrix} \quad (1)$$

Once the matrix \mathbf{A} has been entered, the TI-86 command `eigV1 A` takes only about 15 seconds to calculate the seven eigenvalues shown in the λ -row of the table below. Alternatively, you can use the *Maple* `with(linalg)` command `eigenvals(A)`, the *Mathematica* command `Eigenvalues[A]`, or the MATLAB command `eig(A)`. Then the remaining entries $\omega = \sqrt{-\lambda}$ and $P = 2\pi/\omega$ showing the natural frequencies and periods of oscillation of the seven-story building are readily calculated. Note that a typical earthquake producing ground oscillations with a period of 2 seconds is uncomfortably close to the fifth natural period 1.9869 seconds of the building.

i	1	2	3	4	5	6	7
λ	-38.2709	-33.3826	-26.1803	-17.9094	10.0000	-3.8197	-0.4370
ω	6.1863	5.7778	5.1167	4.2320	3.1623	1.9544	0.6611
P (sec)	1.1057	1.0875	1.2280	1.4847	1.9869	3.2149	9.5042

A horizontal earthquake oscillation $E \cos \omega t$ of the ground, with amplitude E and acceleration $a = -E\omega^2 \cos \omega t$, produces an opposite inertial force $F = ma = mE\omega^2 \cos \omega t$ on each floor of the building. The resulting nonhomogeneous linear system is

$$\mathbf{x}'' = \mathbf{A}\mathbf{x} + (E\omega^2 \cos \omega t)\mathbf{b} \quad (2)$$

where $\mathbf{b} = [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]^T$ and \mathbf{A} is the matrix of Eq. (1). Figure 7.4.18 in the text shows a plot of *maximal amplitude* (for the forced oscillations of any single floor) versus the *period* of the earthquake vibrations. The spikes correspond to the first six of the seven resonance frequencies. We see, for instance, that while an earthquake with period 2 seconds likely would produce destructive resonance vibrations in the building, it probably would be unharmed by an earthquake with period 2.5 seconds. Different buildings have different natural frequencies of vibration, and so a given earthquake may demolish one building, but leave untouched the one next door. This type of apparent anomaly was observed in Mexico City after the devastating earthquake of September 19, 1985.

Investigation

For your personal seven-story building to investigate, let the weight (in tons) of each story equal the largest digit of your student ID number and let k (in tons/ft) equal the smallest nonzero digit. Produce numerical and graphical results like those illustrated in Figs. 7.4.17 and 5.3.18 of the text. Is your building susceptible to likely damage from an earthquake with period in the 2 to 3 second range?

You might like to begin by working manually the following warm-up problems.

1. Find the periods of the natural vibrations of a building with two aboveground floors, with each weighing 16 tons and with each restoring force being $k = 5$ tons/ft.
2. Find the periods of the natural vibrations of a building with three aboveground floors, with each weighing 16 tons and with each restoring force being $k = 5$ tons/ft.
3. Find the natural frequencies and natural modes of vibration of a building with three aboveground floors as in Problem 2, except that the upper two floors weigh 8 tons instead of 16 tons. Give the ratios of the amplitudes A , B , and C of the oscillations of the three floors in the form $A : B : C$ with $A = 1$.
4. Suppose that the building of Problem 3 is subject to an earthquake in which the ground undergoes horizontal sinusoidal oscillations with a period of 3 seconds and an amplitude of 3 inches. Find the amplitudes of the resulting steady periodic oscillations of the three above-ground floors. Assume the fact that a motion $E \cos \omega t$ of the ground, with amplitude E and acceleration $a = -E\omega^2 \cos \omega t$, produces an opposite inertial force $F = ma = mE\omega^2 \cos \omega t$ on a floor of mass m .

A Three-Mass Automobile Model

In the sections that follow, we illustrate appropriate *Maple*, *Mathematica*, and MATLAB techniques by analyzing the natural frequencies of vibration of a car that is modeled by a system of three masses and four springs. Suppose that

- mass m_1 is connected to the chassis by spring k_1 ;
- masses m_1 and m_2 are connected by spring k_2 ;
- masses m_2 and m_3 are connected by spring k_3 ; and
- masses m_1 and m_3 are connected by spring k_4 .

The corresponding linear system $\mathbf{M}\mathbf{x}'' = \mathbf{K}\mathbf{x}$ has coefficient matrices

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad (3)$$

and

$$\mathbf{K} = \begin{bmatrix} -(k_1 + k_2 + k_4) & k_2 & k_4 \\ k_2 & -(k_2 + k_3) & k_3 \\ k_4 & k_3 & -(k_3 + k_4) \end{bmatrix}. \quad (4)$$

The displacement vector $\mathbf{x}(t) = [x_1(t) \ x_2(t) \ x_3(t)]^T$ of the three-mass system then satisfies the equation

$$\mathbf{x}'' = \mathbf{A}\mathbf{x} \quad (5)$$

with coefficient matrix $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$.

We will use the numerical values $m_1 = 40$, $m_2 = 20$, $m_3 = 40$ (in slugs) and $k_1 = 5000$, $k_2 = 1000$, $k_3 = 2000$, $k_4 = 3000$ (in lbs/ft). We want to find the three natural frequencies $\omega_1, \omega_2, \omega_3$ of oscillation of this three-mass system modeling our car. If the car is driven with velocity v (ft/sec) over a washboard surface shaped like a cosine curve with a wavelength of $a = 30$ feet, then the result is a periodic force on the car with frequency $\omega = 2\pi v/a$. We would expect the car to experience resonance vibrations when this forcing frequency equals one of the car's natural frequencies.

Using *Maple*

First we define the masses

```
m1 := 40:    m2 := 20:    m3 := 40:
```