

## Fourier Transforms and the Wave Equation

**Overview and Motivation:** We first discuss a few features of the Fourier transform (FT), and then we solve the initial-value problem for the wave equation using the Fourier transform.

**Key Mathematics:** More Fourier transform theory, especially as applied to solving the wave equation.

### I. FT Change of Notation

In the last lecture we introduced the FT of a function  $f(x)$  through the two equations

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk, \quad (1a)$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \quad (1b)$$

Note that we have changed notation compared to the last lecture. Hereafter we designate the FT of any function by the same symbol, but with an overhead caret included. That is, the FT of  $f(x)$  we now write as  $\hat{f}(k)$ . As we shall see, this is useful when dealing with equations that include FTs of several functions.<sup>1</sup>

### II. Some Properties of the Fourier Transform

We now discuss several useful properties of the Fourier transform.

#### A. Translation

The first property has to do with translation of the function  $f(x)$ . Let's say we are interested in  $f(x-x_0)$ , which corresponds to translation of  $f(x)$  by  $x_0$ . Then, using Eq. (1a) we can write

$$\begin{aligned} f(x-x_0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ik(x-x_0)} dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\hat{f}(k) e^{-ikx_0}] e^{ikx} dk \end{aligned} \quad (2)$$

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<sup>1</sup> This notation is fairly common practice. At some point you may even see the FT of  $f(x)$  written as  $f(k)$ . At least we won't be doing that here!

Thus, we see that the FT of  $f(x - x_0)$  is  $\hat{f}(k)e^{-ikx_0}$ . In other words, translation of  $f(x)$  by  $x_0$  corresponds to multiplying the FT  $\hat{f}(k)$  by  $e^{-ikx_0}$ .

### B. Differentiation

The second property has to do with the FT of  $f'(x)$ , the derivative of  $f(x)$ . Again, using Eq. (1a) we have

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [ik \hat{f}(k)] e^{ikx} dk. \quad (3)$$

So we see that FT of  $f'(x)$  is  $ik \hat{f}(k)$ . That is, differentiation of  $f(x)$  corresponds to multiplying  $\hat{f}(k)$  by  $ik$ .

### C. Integration

Let's consider the definite integral of  $f(x)$ ,

$$\int_{x_1}^{x_2} dx f(x) = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} dx \left[ \int_{-\infty}^{\infty} dk \hat{f}(k) e^{ikx} \right]. \quad (4)$$

Switching the order of integration on the rhs produces

$$\begin{aligned} \int_{x_1}^{x_2} dx f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \hat{f}(k) \left[ \int_{x_1}^{x_2} dx e^{ikx} \right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{\hat{f}(k)}{ik} (e^{ikx_2} - e^{ikx_1}) \end{aligned} \quad (5)$$

So if we define  $If(x)$  to be the indefinite integral of  $f(x)$ , we can rewrite Eq. (5) as

$$If(x_2) - If(x_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{\hat{f}(k)}{ik} (e^{ikx_2} - e^{ikx_1}) \quad (6)$$

So integration of  $f(x)$  essentially corresponds to dividing the Fourier transform  $\hat{f}(k)$  by  $ik$ .<sup>2</sup>

#### D. Convolution

The last property concerning the a function and its FT has to do with convolution. Because you may not be familiar with convolution, let's first define it. Simply put, the convolution of two functions  $f(x)$  and  $g(x)$ , which we denote  $(f * g)(x)$ , is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-x')g(x')dx' \quad (7)$$

Perhaps the most common place that convolution arises is in spectroscopy, where  $g(x)$  is some intrinsic spectrum that is being measured, and  $f(x)$  is the resolution function of the spectrometer that is being used to measure the spectrum.<sup>3</sup> The convolution  $(f * g)(x)$  is the spectrum that is then measured.

Note that  $(f * g)(x)$  is indeed a function of  $x$ , and so we can calculate its FT, which we denote  $(f \hat{*} g)(k)$ . Using Eq. (1b) we can write

$$(f \hat{*} g)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \left[ \int_{-\infty}^{\infty} dx' f(x-x')g(x') \right] e^{-ikx}, \quad (8)$$

which can be rearranged as

$$(f \hat{*} g)(k) = \int_{-\infty}^{\infty} dx' g(x') \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x-x')e^{-ikx} \right]. \quad (9)$$

Now the quantity in brackets is the FT of  $f(x)$  translated by  $x'$ . From Sec. II.B above we know that this is  $\hat{f}(k)e^{-ikx'}$ , and so Eq. (9) can be expressed as

<sup>2</sup> You might think that Eq. (6) could be simplified to  $If(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{\hat{f}(k)}{ik} e^{ikx}$ , but this cannot be done

because indefinite integration produces an undetermined integration constant. The constant does not appear in Eq. (6) because it is an equation for the difference of  $If(x_2)$  and  $If(x_1)$ .

<sup>3</sup> The resolution function is often quite close to a Gaussian of a particular, fixed width.