Worksheet #3: Functions of Several Variables

I. Preliminaries

1. How many independent variables does the equation $z = x^2 - 4y^2$ have?

2. Give the domain and range of the following functions:

   I. $f(x, y) = e^{x^2+y^2}$

   II. $f(x, y) = \sqrt{4x^2 + 4y^2 - 9}$

   III. $f(x, y) = \dfrac{1}{\ln(x + 2y)}$

3. Suppose that $f(x, y) = 2x + y$.

   I. Consider the curve C in the xy-plane defined by $y = x^2$ for $x \geq 0$. Give an expression for the curve $\mathcal{C}$ on the surface $z = f(x, y)$ that lies above C in terms of x only. Sketch both C and the curve on the surface.

   II. Give a parametric description $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ of the curve $\mathcal{C}$.

   III. Explain whether the point $(x, y, z) = (2, 4, 10)$ lies on $\mathcal{C}$.

II. Planes

4. Explain how to find the equation of a plane with normal vector $\vec{n} = \langle a, b, c \rangle$ that passes through a point (x_0, y_0, z_0).

5. Consider the plane $3x + y - z = 2$.

   I. Give a normal vector $\vec{n}$ for the plane.

   II. A. Show that the points $(2, 0, 4)$ and $(0, 1, -1)$ lie on the plane.

   B. Find a vector $\vec{v}$ whose initial point is $(2, 0, 4)$ and whose terminal point is $(0, 1, -1)$. Verify that $\vec{v}$ is orthogonal to the vector $\vec{n}$ you found in Part I.

   III. Suppose the points (x_0, y_0, z_0) and (x_1, y_1, z_1) lie on the plane. Find a vector $\vec{u}$ whose initial point is (x_0, y_0, z_0) and whose terminal point is (x_1, y_1, z_1). Verify that $\vec{u}$ is orthogonal to the vector $\vec{n}$ you found in Part I.

   This provides justification why the vector $\vec{n}$ you found in Part I is indeed normal to the plane!

6. Explain how to determine whether two planes are parallel or perpendicular.

7. Determine whether each collection of planes P_1 and P_2 below are parallel, perpendicular, or neither.

   I. P_1: $2x + 3y - z = 4$, P_2: $x - 2y - 4z = 5$

   II. P_1: $x - 7y - 3z = 6$, P_2: $2x - 14y - 6z = 5$

   III. P_1: $x - 2y + z = 5$, P_2: $2x + y - z = 4$

8. Find the equation of the plane parallel to $2x - 3y + z = 6$ that passes through $(1, 2, 3)$. Express your final answer in the form $ax + by + cz = d$.

9. Is there a unique plane that passes through the points $(1, 0, 2)$ and $(-1, 1, 0)$? If there is not, what is the most general form of a plane that passes through them?

10. Find the equation of a plane that passes through $(0, 1, 0), (2, 1, 0),$ and $(-1, 1, 1)$.

11. Consider the planes P_1: $2x - 3y + z = 1$, P_2: $5x + y + 2z = 0$.

    I. Verify that the planes P_1 and P_2 are not parallel.

    II. Find a parametric description of the line of intersection of the planes.

    III. Use your parametric description to find a point (x, y, z) on the line by choosing a specific value for the parameter. Verify that this point lies on both P_1 and P_2.

    IV. Verify that your parametric description $\vec{r}(t)$ lies on both planes.

III. Paths and Level Curves

12. Suppose a surface $\mathcal{S}$ in the xyz-plane is described by the function $z = f(x, y)$. Given a path C in the xy-plane that lies in the domain of $f(x, y)$, explain how to find a parameterization of the curve on the surface $\mathcal{S}$ that lies above the path C.

13. Suppose that the surface $\mathcal{S}$ is described by the function $f(x, y) = x^2 + 3y$. Give a parameterization of the curve on $\mathcal{S}$ that lies above the following paths in the xy-plane:

    I. $y = 3x^2$ II. $2x + 3y = 4$ III. $x^2 + y^2 = 9$

14. Explain how to find the level curves of a surface described by $z = f(x, y)$.

15. Suppose that $z = f(x, y)$ is a surface and it is known that when $z = 2$, the corresponding level curve is $x^2 + y = 7$.

    I. Explain, using words only, what this means.

    II. Give a parametrization of the contour curve in the xyz-plane corresponding to this curve.

    III. Sketch the level curve in the xy-plane and the corresponding contour curve in the xyz-plane.

16. Suppose that $z = x^2 + y^2$ is a surface.

    I. Find and classify the level curves for this surface.

    II. Sketch the level curves corresponding to $z = 1$, $z = 4$, and $z = 9$ in the xy-plane and the corresponding contour curves in the xyz-plane. Use this to sketch the surface.

    III. Find a parameterization for both the level curve corresponding to $z = 4$ and the corresponding contour curve.

17. Suppose that $z = x + y^2$ is a surface.

    I. Find the level curve that passes through the point $(2, 3)$ in the xy-plane in terms of x and y.

    II. Find a parameterization for the level curve and the actual contour curve $z = 2$.

    III. Sketch the level curve in the xy-plane and the corresponding contour curve in the xyz-plane.

18. Consider the function $f(x, y) = \dfrac{x - y^3}{2x + y^3}$.

    I. State the domain of the function.

    II. Suppose C is a curve in the xy-plane for which $\vec{r}(t) = \langle 2t^3, t \rangle$, $t > 0$ is a parameterization. Determine whether C lies on a level curve of $f(x, y)$. If it does, give a parameterization of the portion of the contour curve to which it corresponds.

19. Consider the function $f(x, y) = \dfrac{x - y}{x^2 + y^2}$.

    I. State the domain of the function.

    II. Suppose C is a curve in the xy-plane for which $\vec{r}(t) = \langle t, t^2 \rangle$, $t > 0$ is a parameterization. Determine whether C lies on a level curve of $f(x, y)$. If it does, give a parameterization of the portion of the contour curve to which it corresponds.

IV. Limits

20. Explain what it means conceptually to say that $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$.

21. Explain several methods from which it can be determined that $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ does not exist.

22. Consider the function $f(x, y) = \dfrac{6xy}{5x^2 + 4y^2}$.

    I. A student provides the response to the question indicated below. Determine if the response is correct or incorrect.