8/17/2004 FINAL EXAM Maths 21a, O. Knill, Summer 2004

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points)

T	F	
T	F	The distance of the point $(3,1,6)$ and the point $(1,1,1)$ is $\sqrt{29}$.

Solution:
True. By definition.

T	F	
T	**F**	$\int_0^2 \int_0^{2\pi} (r^2/2)\, d\theta\, dr$ computes the area of a disc of radius 2 in the plane.

Solution:
False. The correct formula is $\int_0^{2\pi} \int_0^2 r\, dr\, d\theta$

T	F	
T	**F**	The identity $\vec{v} \cdot \vec{w} \geq |\vec{v}| \cdot |\vec{w}|$ is always true.

Solution:
False. The inequality is the other way round.

T	F	
T	**F**	The velocity and acceleration vectors of a curve $\vec{r}(t) = (x(t), y(t))$ are always perpendicular.

Solution:
False. This is already false for a line $r(t) = (t^2, t^2)$, where the velocity and acceleration are parallel.

T	F	
T	F	A circle of radius 5 has a smaller curvature than a circle of radius 1.

Solution:
True. The curvature of a circle of radius r is equal to $1/r$.

T	F	
T	F	The curve $\vec{r}(t) = (-\sin(t), \cos(t))$ for $t \in [0, 2\pi]$ is a circle.

Solution:
True. Indeed, one can check that $\sin^2(t) + \cos^2(t) = 1$.

T **F** The function $\sin(x-t)$ is a solution of the Burger equation $u_t = uu_x + u_{xx}$.

Solution:
False, $uu_x = \sin(x-t)\cos(x-t)$, $u_t = -\cos(x-t)$ and $u_{xx} = -\sin(x-t)$. For $t=0$, the two sides are different.

T **F** The length of the curve $\vec{r}(t) = (t^3, t^2)$ on $[1,2]$ is $\int_1^2 (9t^4 + 4t^2)\, dt$.

Solution:
False. The derivatives have to be squared. The correct answer would be $\int_1^2 \sqrt{9t^4 + 4t^2}\, dt$.

T **F** Let (x_0, y_0) be the maximum of $f(x,y)$ under the constraint $g(x,y) = 1$. Then the gradient of g at (x_0, y_0) is perpendicular to the gradient of f at (x_0, y_0).

Solution:
The gradients are **parallel**, not perpendicular.

T **F** The directional derivative $D_{\vec{v}} f(x_0, y_0, z_0)$ of $f(x,y,z) = x^2 + y^2 - z^2$ into the direction $\vec{v} = (0,0,1)$ is negative at every point (x_0, y_0, z_0).

Solution:
This directional derivative is $f_z = -2z$. For $z > 0$, this is negative, for $z < 0$, this is positive.

T F If a vector field $F(x,y)$ is not conservative, we always can find a curve C for which the line integral $\int_C F \cdot dr$ is positive.

Solution:
Indeed. There must exist then a curve for which the line integral is not zero. If this line integral is positive, we have found our curve, if it is negative, we reverse the direction of the curve.

T F If C is a closed level curve of a function $f(x,y)$ and $F = (f_x, f_y)$ is the gradient field of f, then $\int_C F \cdot dr = 0$.

Solution:
The gradient field is perpendicular to the level curves.

T **F** The divergence of the gradient of any function $f(x,y,z)$ is always zero.

Solution:
No, just take a simple example like $f(x,y,z) = x^2$, where $\mathrm{div}(\mathrm{grad}(f)) = 2$.

T **F** The function $f(x,y) = x^2 y^3$ has no critical points.

Solution:
$\nabla f(x,y) = (2xy^3, 3x^2y^2)$, which vanishes at $(0,0)$.

T F If $F(x,y) = (y, 2x)$ and $C : \vec{r}(t) = (\sqrt{\cos(t)}, \sqrt{\sin(t)})$ parameterizes the boundary of the region $R : x^4 + y^4 \leq 1$, then $\int_C F \cdot dr$ is the area of R.

Solution:
This is a direct consequence of Green's theorem and the fact that the two-dimensional curl $Q_x - P_y$ of $F = (P, Q)$ is equal to 1.

T F The flux of the vector field $F(x,y,z) = (0, y, 0)$ through the boundary S of a solid torus E is equal to the volume of the torus.

Solution:
It is the **volume** of the solid torus.

T F The quadratic surface $x^2 + y^2 + 4x - z^2 = -3$ is a one sheeted hyperboloid.

Solution:
Completion of the square gives the equation $(x+2)^2 + y^2 - z^2 = 1$.

T F If F is a vector field in space and S is the boundary of a solid torus, then the flux of $\mathrm{curl}(\vec{F})$ through S is 0.

Solution:
This is true by Stokes theorem.

T	F

If $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) and S is a torus surface, then the flux of F through S is zero.

Solution:
This is a consequence of the divergence theorem.

T	F

In spherical coordinates, the equation $\rho\cos(\phi) = \rho\sin(\theta)\sin(\phi)$ defines a plane.

Solution:
True. It is the plane $z = y$.

Problem 2)	(10 points)